Two-Dimensional Frequency Domain Analysis of String Stability

Steffi Knorn and Richard H. Middleton

Abstract— **String stability issues in vehicle platoons have been studied in various ways. In the homogeneous unidirectional case, the platoon can be considered as a two dimensional (2D) continuous-discrete systems with an unavoidable singularity on the stability boundary and care is needed in their analysis. Frequency domain analysis of such 2D systems allows analysis of BIBO stability and other features.**

I. Introduction

One control objective in the field of coordinated systems is formation control. Here a group of vehicles (e.g. platoon) should follow a given group trajectory and in addition every vehicle needs to maintain a prescribed distance to the surrounding vehicles.

In its simplest form platoon control requires a constant distance between the vehicles and the lead vehicle to follow a given trajectory, e.g. [1]–[3]. To simplify communication requirements we consider the case where the automobiles are equipped with a local controller based on sensing the distance to the preceding vehicle. We call the string *homogeneous* if the dynamics of the vehicle and controller are independent of location in the string.

Even though it is easy to achieve a stable string in the conventional understanding, i.e. the local error signals for every vehicle in the string are bounded and go to zero, error signals can amplify when traveling through the string resulting in the local error norm to grow with the position in the string. This effect is referred to as 'string instability' or 'slinky effect'. It has been shown that it is not possible to achieve string stability in a homogeneous string of strictly proper feedback control systems with nearest neighbour communications when using only linear systems with two integrators in the open loop and constant inter-vehicle spacing, [3], [4], independent of the particular linear controller design, [5]. However, string stability can be guaranteed among other things with a speed dependent inter-vehicle spacing policy (also called 'time headway policy'), [6].

In the past, different definitions of string stability have been used. While most researchers work with input-output formulations, definitions involving the initial conditions and state space formulations can also be found, e.g. [5], [7]–[9].

One possibility to discus string stability is to treat the platoon as a two-dimensional continuous-discrete system with two independent variables: continuous time t and the discrete position within the string k.

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Due to a wider field of applications the related field of two-dimensional discrete systems has been studied in more detail. One of the first sufficient conditions for Bounded-Input Bounded-Output (BIBO) stability in the frequency domain (using the two-dimensional Z transform and requiring the poles of the system transfer function to lay inside the open stability region) was presented in [10], leading to different stability tests such as in [11], [12].

Similar results for BIBO stability of continuous systems (using the two-dimensional Laplace transform) can be found in [13] and stability tests in [14]–[16].

Most researchers, however, explicitly or implicitly, exclude an important special case called Nonessential Singularity of the Second Kind (NSSK) where there exist a set of (z_1,z_2) (in the discrete case), (s_1,s_2) (in the continuous case) or s,z (in the continuous-discrete case) such that both the denominator and the numerator of the transfer function go to zero at the same time. It was shown that while some transfer functions with an NSSK on the stability boundary are BIBO stable, others with an NSSK at the same place are BIBO unstable, [17]. This was followed by a sufficient stability condition in [18] and a necessary condition that the system cannot be stable if the NSSK lies outside the boundary of the stability region, [19].

Here we will use the combined Laplace-Z transform introduced in [20] to analyse the two-dimensional continuous-discrete platooning problem in the frequency domain. Due to the nature of that system, an NSSK at the stability boundary, i.e. at $s = 0$ and $z = 1$, cannot be avoided. Thus, the BIBO stability of the system cannot be determined by the location of the poles of the transfer function. In fact, to guarantee BIBO stability, it is sufficient to show that the induced operator norm is bounded in the stability region.

We will start with preliminary results in Section II, discuss the induced operator norm in Section III and analyse the stability of a simple platooning problem in Section IV.

II. Mathematical Preliminaries

We assume that $x(t,k)$ is a two dimensional continuous-discrete signal, which does not grow faster than exponentially, i.e. $\exists c,a,b,T,K < \infty$ such that

$$|x(t,k)| \le c\mathrm{e}^{at}b^k \qquad \text{for all } t \ge T \text{ and } k \ge K. \tag{1}$$

Thus, the unilateral, combined Laplace-Z transform $X(s,z)$ of $x(t,k)$ is defined as, [20],

$$X(s,z) = \mathcal{Z}\mathcal{L}\{x(t,k)\} = \mathcal{Z}\{\ \mathcal{L}\{x(t,k)\}\} = \sum_{k=0}^{\infty}\int_0^{\infty} x(t,k)\mathrm{e}^{-st}\mathrm{d}tz^{-k} \tag{2}$$

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and its inverse

$$x(t,k) = \{\mathcal{ZL}\}^{-1}\{X(s,z)\} = \frac{1}{(2\pi j)^2}\oint_C \int_{\alpha-j\infty}^{\alpha+j\infty} X(s,z)\mathrm{e}^{st}\mathrm{d}s z^{k-1}\mathrm{d}z$$

where $\alpha > a$ and C is the contour $|z| = \beta > b$. In the case where $a < 0$ and $b < 1$ (that is where the two-dimensional transform has no poles with $\Re(s) \geq 0$, and $|z| \geq 1$) we can take $\alpha = 0$ and $\beta = 1$, such that

$$x(t,k) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} X\left(j\omega, \mathrm{e}^{j\theta}\right)\mathrm{e}^{j\omega t}\mathrm{d}\omega \mathrm{e}^{j\theta k}\mathrm{d}\theta \qquad (3)$$

To prove Parseval's Theorem later several properties of the Laplace-Z transformation are needed (some are given in [20], others are simple extensions of well known results on Laplace and Z transform, [21]–[23]):

Permutability: Assuming that both transforms and inverse transforms exist, we can write $X(s,z) = \mathcal{ZL}\{x(t,k)\} = \mathcal{Z}\{\mathcal{L}\{x(t,k)\}\} = \mathcal{L}\{\mathcal{Z}\{x(t,k)\}\}$ and $x(t,k) = \{\mathcal{ZL}\}^{-1}\{X(s,z)\} = \mathcal{Z}^{-1}\left\{\mathcal{L}^{-1}\{X(s,z)\}\right\} = \mathcal{L}^{-1}\left\{\mathcal{Z}^{-1}\{X(s,z)\}\right\}$.

Integration: If the Laplace-Z transform of $x(t,k)$ is $X(s,z)$ and the integral $\int_0^t x(\tau,k)\mathrm{d}\tau$ exists, we can write

$$\mathcal{ZL}\left\{\int_0^t x(\tau,k)\mathrm{d}\tau\right\} = \frac{1}{s}X(s,z). \qquad (4)$$

Accumulation: If the Laplace-Z transform of $x(t,k)$ is $X(s,z)$, then the Laplace-Z transform of the cumulative sum can be written as

$$\mathcal{ZL}\left\{\sum_{i=0}^{k} x(t,i)\right\} = \frac{1}{1-z^{-1}}X(s,z). \qquad (5)$$

Final Value Theorem: If the final values $\lim_{t\to\infty} x(t,k)$ and $\lim_{k\to\infty} x(t,k)$ exist, then

$$\lim_{(t,k)\to(\infty,\infty)} x(t,k) = \lim_{(s,z)\to(0,1)} s(1-z^{-1})X(s,z). \qquad (6)$$

Multiplication: If both Laplace-Z transforms of $x_1(t,k)$ and $x_2(t,k)$ exist ($X_1(s,z)$ and $X_2(s,z)$, respectively), the Laplace-Z transform of $x_1(t,k)x_2(t,k)$ is a combined convolution in the frequency domain

$$\mathcal{ZL}\{x_1(t,k)x_2(t,k)\}$$
$$= \frac{1}{(2\pi j)^2}\oint_C \int_{\alpha_1-j\infty}^{\alpha_1+j\infty} X_1(p,v)X_2\left(s-p,\frac{z}{v}\right)v^{-1}\mathrm{d}p\mathrm{d}v. \qquad (7)$$

Lemma 1 (Parseval's Theorem for 2D Cont.-Disc. Sys.): If the Laplace-Z transform $X(s,z)$ of $x(t,k)$ exists and there exist $a < 0$ and $|b| < 1$ such that $|x(t,k)| \leq c\mathrm{e}^{at}b^k$, the L_2-norm in the time domain is bounded and the same as the L_2-norm in the frequency domain:

$$\sum_{k=0}^{\infty}\int_0^{\infty} x^2(t,k)\mathrm{d}t = \|x(\cdot,\cdot)\|_2^2 = \|X(\cdot,\cdot)\|_2^2$$
$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} X^2\left(j\omega,\mathrm{e}^{j\theta}\right)\mathrm{d}\omega\mathrm{d}\theta. \qquad (8)$$

Proof: First, we will define $\phi(t,k)$, $\psi(t,k)$, and $\xi(t,N)$ such that

$$\int_0^t x^2(\tau,k)\mathrm{d}\tau = \int_0^t \phi(\tau,k)\mathrm{d}\tau = \psi(t,k), \quad \sum_{k=0}^{N}\psi(t,k) = \xi(t,N). \qquad (9)$$

Since $x(t,k) \in L_2\ [0,\infty)\times[0,\infty)$ and $x(t,k) \in \mathbb{R}$

$$\sum_{k=0}^{N}\int_0^t x^2(t,k)\mathrm{d}t \leq \sum_{k=0}^{\infty}\int_0^{\infty} x^2(t,k)\mathrm{d}t = \sum_{k=0}^{\infty}\int_0^{\infty}|x(t,k)|_2^2\mathrm{d}t < \infty \qquad (10)$$

Thus, the order of summation, integration and limits can be interchanged. We can write the norm of $x(t,k)$ as

$$\|x(\cdot,\cdot)\|_2^2 = \sum_{k=0}^{\infty}\int_0^{\infty} x^2(t,k)\mathrm{d}t = \sum_{k=0}^{\infty}\lim_{t\to\infty}\int_0^t x^2(\tau,k)\mathrm{d}\tau$$
$$= \lim_{t\to\infty}\sum_{k=0}^{\infty}\psi(t,k) = \lim_{t\to\infty}\lim_{N\to\infty}\xi(t,N) \qquad (11)$$

With the final value theorem (6), the limit of $\xi(t,N)$ in (11) can be expressed as the limit in the frequency domain of the corresponding Laplace-Z transform $\Xi(s,z)$,

$$\|x(\cdot,\cdot)\|_2^2 = \lim_{s\to 0}\lim_{z\to 1} s\left(1-z^{-1}\right)\Xi(s,z) \qquad (12)$$

Because $\xi(t,N)$ is the accumulation of $\psi(t,k)$, (9), (12) yields

$$\|x(\cdot,\cdot)\|_2^2 = \lim_{s\to 0}\lim_{z\to 1} s\Psi(s,z) \qquad (13)$$

where $\Psi(s,z) = \mathcal{LZ}\{\psi(t,k)\}$. Since $\psi(t,k)$ is the integral of $\phi(t,k)$, (9), we have $\mathcal{LZ}\{\phi(t,k)\} = \Phi(s,z) = s\Psi(s,z)$ and can write according to (4)

$$\|x(\cdot,\cdot)\|_2^2 = \lim_{s\to 0}\lim_{z\to 1}\Phi(s,z) \qquad (14)$$

Furthermore, we know that a multiplication in the time domain corresponds to a convolution in the frequency domain, (7), and transform (14) into

$$\|x(\cdot,\cdot)\|_2^2 = \lim_{s\to 0}\lim_{z\to 1}\frac{1}{(2\pi j)^2}\oint_C\int_{c-j\infty}^{c+j\infty} X(p,v)X\left(s-p,\frac{z}{v}\right)v^{-1}\mathrm{d}p\mathrm{d}v$$
$$= \frac{1}{(2\pi j)^2}\oint_C\int_{c-j\infty}^{c+j\infty}|X(s,z)|^2 z^{-1}\mathrm{d}s\mathrm{d}z \qquad (15)$$

Since $X(s,z)$ does not have any poles with $\Re(s) \geq 0$, and $|z| \geq 1$ (that is the same as requiring $|x(t,k)| \leq c\mathrm{e}^{at}b^k$ with $a < 0$ and $|b| < 1$), (15) becomes

$$\|x(\cdot,\cdot)\|_2^2 = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|X\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta$$
$$= \|X(\cdot,\cdot)\|_2^2 \qquad (16)$$

Thus, the Euclidean norm in time domain is equivalent to the Euclidean norm in the frequency domain $\|x(\cdot,\cdot)\|_2 = \|X(\cdot,\cdot)\|_2$. ■

III. Induced Operator Norm

We now want to find the induced 2-norm for the norm in ω and θ introduced in (16). Consider the two-dimensional continuous-discrete system $\hat{E}\left(j\omega,e^{j\theta}\right) = H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\hat{D}\left(j\omega,e^{j\theta}\right)$. The induced 2-norm of $H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)$ is the upper bound for the norm of $\hat{E}\left(j\omega,e^{j\theta}\right)$ for all $\hat{D}\left(j\omega,e^{j\theta}\right)$ with $\|\hat{D}(\cdot,\cdot)\|_2 = 1$. Assume $H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)$ is continuous almost everywhere except at a finite number of discontinuous "pinch off" points at the NSSKs $\left(j\omega_p,e^{j\theta_p}\right)$. For every such point there exist a curve $\theta = \theta(\omega)$ and a function $g(\omega) = \left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta(\omega)}\right)\right|$ such that $\lim_{\omega\to\omega_p} g(\omega) = C$. Then the induced operator norm of $H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)$ is

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{i_2} = \operatorname*{ess\,sup}_{\omega,\theta}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|. \tag{17}$$

The induced Norm of $H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)$ is defined as

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{i_2}^2 := \sup_{\|\hat{D}\|_2^2=1}\left\|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\hat{D}\left(j\omega,e^{j\theta}\right)\right\|_2^2$$
$$= \sup_{\|\hat{D}\|_2^2=1}\left(\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\hat{D}\left(j\omega,e^{j\theta}\right)\right|^2 d\omega d\theta\right). \tag{18}$$

First, we will show that the essential supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|$ over all ω and θ is the upper bound of the induced operator norm. With Hölder's Inequality, [24, Theorem 3.8.2], (18) yields

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{i_2}^2 \le \operatorname*{ess\,sup}_{\omega,\theta}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|^2$$
$$\cdot\sup_{\|\hat{D}\|_2^2=1}\left(\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|\hat{D}\left(j\omega,e^{j\theta}\right)\right|^2 d\omega d\theta\right)$$
$$= \operatorname*{ess\,sup}_{\omega,\theta}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|^2. \tag{19}$$

To show that the essential supremum also is a lower bound we will use the following Lemma:

Lemma 2: Given a two-dimensional operator $H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)$ which is continuous in $\left(j\omega_0,e^{j\theta_0}\right)$ the induced operator norm of $H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)$ is always greater or equal to the magnitude of $H_{\hat{e},\hat{d}}\left(j\omega_0,e^{j\theta_0}\right)$:

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{i_2} \ge \left|H_{\hat{e},\hat{d}}\left(j\omega_0,e^{j\theta_0}\right)\right|. \tag{20}$$

Proof: We choose the disturbance signal

$$\hat{d}_\epsilon(t,k) = \alpha_{\omega_0}e^{-\epsilon t}\cos\omega_0 t\cdot\alpha_{\theta_0}e^{-\epsilon k}\cos\theta_0 k \tag{21}$$

with

$$\alpha_{\omega_0}^2 = \frac{4\epsilon^2+4\omega_0^2}{2\epsilon+\omega_0^2/\epsilon} \tag{22}$$

and

$$\alpha_{\theta_0}^2 = \frac{2}{\frac{1}{1-e^{-2\epsilon}}+\frac{1-e^{-2\epsilon}\cos 2\theta_0}{1-2e^{-2\epsilon}\cos 2\theta_0+e^{-4\epsilon}}} \tag{23}$$

to guarantee $\left\|\hat{d}_\epsilon(\cdot,\cdot)\right\|_2 = 1$. We will now use the following trick with $\mathcal{R}_\epsilon = \{\omega,\theta : |\omega-\omega_0| \le \sqrt{\epsilon}, |\theta-\theta_0| \le \sqrt{\epsilon}\}$.

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{i_2}^2 = \operatorname*{ess\,sup}_{\|\hat{D}\|_2=1}\left\|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\hat{D}\left(j\omega,e^{j\theta}\right)\right\|_2^2$$
$$\ge \left\|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\hat{D}_\epsilon\left(j\omega,e^{j\theta}\right)\right\|_2^2$$
$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|^2\left|\hat{D}_\epsilon\left(j\omega,e^{j\theta}\right)\right|^2 d\omega d\theta$$
$$\ge \frac{1}{(2\pi)^2}\iint_{\omega,\theta\in\mathcal{R}_\epsilon}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|^2\left|\hat{D}_\epsilon\left(j\omega,e^{j\theta}\right)\right|^2 d\omega d\theta$$
$$\ge \frac{1}{(2\pi)^2}\operatorname*{ess\,inf}_{\omega,\theta\in\mathcal{R}_\epsilon}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|^2\iint_{\omega,\theta\in\mathcal{R}_\epsilon}\left|\hat{D}_\epsilon\left(j\omega,e^{j\theta}\right)\right|^2 d\omega d\theta$$

To conclude the proof we will now take the limit for ϵ approaching 0. Given that $H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)$ is continuous around ω_0 and θ_0 the limit of $\operatorname{ess\,inf}_{\mathcal{R}_\epsilon}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|$ is

$$\lim_{\epsilon\to 0}\operatorname*{ess\,inf}_{\mathcal{R}_\epsilon}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right|^2 = \left|H_{\hat{e},\hat{d}}\left(j\omega_0,e^{j\theta_0}\right)\right|^2$$

To evaluate the integral of $\left|\hat{D}_\epsilon\left(j\omega,e^{j\theta}\right)\right|^2$ over $\mathcal{R}_\epsilon$ requires some more work. We will use the Laplace-Z transform of $\hat{d}_\epsilon(t,k)$ and show that in the limit $\epsilon\to 0$ the integral is equal to $(2\pi)^2$. For details refer to Section VI. Note that for $\omega_0 = 0$ or $\theta_0 = 0$ a simplified $d_\epsilon(t,k)$ can be chosen with $\hat{d}_{\epsilon_\omega}(t) = \alpha_\omega e^{-\epsilon t}$ or $\hat{d}_{\epsilon_\theta}(k) = \alpha_\theta e^{-\epsilon k}$, respectively. The corresponding coefficients are $\alpha_\omega^2 = 2\epsilon$ and $\alpha_\theta^2 = 1-e^{-2\epsilon}$. It can be shown in the same way that the limit of $\int_{\omega,\theta\in\mathcal{R}_\epsilon}\left|\hat{D}_\epsilon(j\omega,e^{j\theta})\right|^2 d\omega d\theta$ is $(2\pi)^2$. Thus,

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{i_2} \ge \left|H_{\hat{e},\hat{d}}\left(j\omega_0,e^{j\theta_0}\right)\right|.$$

■

If the essential supremum of $\left|H_{\hat{e},\hat{d}}\right|$ exists it can be achieved in three different cases:

First we will assume that the essential supremum of $\left|H_{\hat{e},\hat{d}}\right|$ is achieved at $\overline{\omega}$ and $\overline{\theta}$ and $\left|H_{\hat{e},\hat{d}}\right|$ is continuous in and around the supremum. In that case, we can set $(\omega_0,\theta_0) = (\overline{\omega},\overline{\theta})$ in (20) and use Lemma 2.

However, it is also possible that the essential supremum is achieved at a discontinuous point (ω_p,θ_p) of $\left|H_{\hat{e},\hat{d}}\right|$. We will use the assumptions made at the beginning of this section. For every discontinuous "pinch off" point there exist a curve $\theta = \theta(\omega)$ and a function $g(\omega) = \left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta(\omega)}\right)\right|$ such that $\lim_{\omega\to\omega_p} g(\omega) = C$. We assume that $C = \operatorname{ess\,sup}_{\omega,\theta}\left|H_{\hat{e},\hat{d}}\right|$. Using the Lemma above we can show that $\|H_{\hat{e},\hat{d}}\|_{i_2} \ge g(\omega)$ arbitrarily close to (ω_p,θ_p). Therefore, it most be true that $\|H_{\hat{e},\hat{d}}\|_{i_2} \ge C$.

In the third case the supremum of $\left|H_{\hat{e},\hat{d}}\right|$ is achieved at $\omega_0\to\infty$ and θ_0

$$\operatorname*{ess\,sup}_{\omega,\theta}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta}\right)\right| = \lim_{\omega\to\infty}\left|H_{\hat{e},\hat{d}}\left(j\omega,e^{j\theta_0}\right)\right|.$$

For the time dependent part of $\hat{d}_\epsilon(t,k)$ we will choose $\epsilon = 1/N$ and $\omega_0 = N$ so that $d_N(t) = \alpha_N e^{-t/N}\cos Nt$ with

$\alpha_N^2 = 4\frac{1+N^4}{2N^2+N^5}$. It can then be shown that for $N \to \infty$ the integral of $|\hat{D}_N(\omega)|^2$ for $\omega \in \left[N - N^{-1/2}, N + N^{-1/2}\right]$ is equal to 2π. At the same time we can use a similar argument as above to show that $\|H_{\hat{e},\hat{d}}\|_{\mathrm{i}_2} \geq \lim_{\omega\to\infty} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta_0}\right)\right|$.

Thus, it is always true that

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{\mathrm{i}_2} \leq \operatorname*{ess\,sup}_{\omega,\theta} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right| \tag{24}$$

Together with (19) the induced L_2-norm of $H_{\hat{e},\hat{d}}(j\omega, \mathrm{e}^{j\theta})$ is

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{\mathrm{i}_2} = \left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{\infty} := \operatorname*{ess\,sup}_{\omega,\theta} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right| \tag{25}$$

IV. Linear, Unidirectional Control

A. System Description

We wish to discuss the stability of a simple chain of vehicles where all but the first should keep a fixed distance $\hat{x}_\mathrm{d}$ to their predecessor. The first car follows a given trajectory. We will choose the same vehicle model with transfer function $P(s)$ and the same linear controller $C(s)$ for every car. The open loop transfer function $L(s)$ has exactly two poles at the origin, $L(s) = P(s)C(s) = \frac{1}{s^2}\tilde{L}(s)$ with $\tilde{L}(0) \neq 0$. The position of the kth vehicle $\hat{x}(t,k)$ depends on the disturbance $\hat{d}(t,k)$ and the actuator signal of the kth controller $\hat{u}(t,k)$. The local control objective is to force the separation error $\hat{e}(t,k)$ to zero. Measurement noise is neglected. Using the Laplace transform with respect to time t the system is described by

$$\hat{X}(s,k) = P(s)\left(\hat{U}(s,k) + \hat{D}(s,k)\right) \tag{26}$$

$$\hat{U}(s,k) = C(s)\hat{E}(s,k) \tag{27}$$

$$\hat{E}(s,k) = \hat{X}(s,k-1) - \hat{X}(s,k) - \frac{\hat{x}_\mathrm{d}}{s} \tag{28}$$

It is known that the absolute value of the complementary sensitivity function of a single subsystem, $T(s) = \frac{L(s)}{1+L(s)}$, is greater than one for a range of frequencies $\omega \in (\omega_-, \omega_+)$, and that the system therefore will be 'string unstable' for constant spacing ($\hat{x}_\mathrm{d} = \mathrm{const}$), [3], [5], which means that the error signals $\hat{e}(t,k)$ grow without bound as k increases.

Since using a constant spacing policy the system is string unstable, a linear time headway h is incorporated in the feedback path. In addition to a fixed vehicle separation, a velocity $\hat{v}(t,k)$ dependent distance is required between the vehicles, $\hat{x}_\mathrm{d}(t,k) = \hat{x}_{\mathrm{d}_0} + h\hat{v}(t,k)$.

Note that in order to preserve the closed loop poles of the time headway free system, an additional pole is inserted into the controller transfer function such that $C_\mathrm{h}(s) = \frac{C(s)}{Q(s)}$ with $Q(s) = hs + 1$.

To simplify the following derivations and because we are interested in the disturbance to error behaviour we shall set $\hat{x}_{\mathrm{d}_0} = 0$ below. The dependence of the error signal $\hat{e}(t,k)$ on the disturbances $\hat{d}(t,k)$ and $\hat{d}(t,k-1)$ and on the previous error $\hat{e}(t,k-1)$ for $k > 1$ can be expressed as

$$\begin{aligned}\hat{E}(s,k) &= \hat{X}(s,k-1) - Q(s)\hat{X}(s,k)\\ &= \Gamma(s)\hat{E}(s,k-1) + \Gamma(s)C_\mathrm{h}^{-1}(s)\left(\hat{D}(s,k-1) - Q(s)\hat{D}(s,k)\right)\end{aligned} \tag{29}$$

with the single loop complementary sensitivity function

$$\Gamma(s) = \frac{P(s)C_\mathrm{h}(s)}{1 + Q(s)P(s)C_\mathrm{h}(s)} = \frac{1}{Q(s)}\frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{T(s)}{Q(s)}.$$

Applying the Z transform (29) becomes

$$\begin{aligned}\hat{E}(s,z) &= \Gamma(s)z^{-1}E(s,z) + \Gamma(s)C_\mathrm{h}^{-1}(s)\left(z^{-1} - Q(s)\right)\hat{D}(s,z)\\ &= \underbrace{\frac{z^{-1} - Q(s)}{1 - z^{-1}\Gamma(s)}\Gamma(s)C_\mathrm{h}^{-1}(s)}_{H_{\hat{e},\hat{d}}(s,z)}\hat{D}(s,z)\end{aligned} \tag{30}$$

with $\hat{E}(s,z) = \mathcal{Z}\left\{\hat{E}(s,k)\right\} = \mathcal{Z}\mathcal{L}\{\hat{e}(t,k)\}$ and $\hat{D}(s,z) = \mathcal{Z}\left\{\hat{D}(s,k)\right\} = \mathcal{Z}\mathcal{L}\left\{\hat{d}(t,k)\right\}$.

B. Conditions for String Stability

To guarantee $\left\|\hat{E}(s,z)\right\|_2 < \infty$ for any $\hat{D}(s,z)$ satisfying $\left\|\hat{D}(s,z)\right\|_2 < \infty$, $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ must be bounded for any ω and θ.

This is always true if $H_{\hat{e},\hat{d}}(s,z)$ has no poles with $\{\Re\{s\} \geq 0\} \cap \{|z| \geq 1\}$. As discussing stability of the string only makes sense for strings with stable subsystems $\Gamma(s)$ must not have any poles with $\Re\{s\} \geq 0$. Also a local controller with zeros with positive real parts has to be avoided to guarantee $\left|C_\mathrm{h}^{-1}(s)\right| < \infty$ for $\Re\{s\} \geq 0$.

Note that $\hat{X}(s,k) = \Gamma(s)\hat{X}(s,k-1) + \Gamma(s)C_\mathrm{h}^{-1}(s)\hat{D}(s,k)$. Every vehicle should be able to follow its predecessor and the local error should be forced to 0 for $t \to \infty$. Therefore, the subsystem closed loop transfer function $\Gamma(s)$ is designed such that $\Gamma(0) = 1$. However, this implies that $H_{\hat{e},\hat{d}}(s,z)$ will always have a pole at $\{s = 0\} \cap \{z = 1\}$. Note that the numerator of $H_{\hat{e},\hat{d}}(s,z)$ is also 0 at the same point, i.e. $1 - Q(0) = 1 - 1 - h\cdot 0 = 0$. This is referred to as a nonessential singularity of the second kind (NSSK).

Therefore, we have to show that $\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ is bounded.

$$\begin{aligned}&\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|\\ &= \lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{\mathrm{e}^{-j\theta} - Q(j\omega)}{1 - \mathrm{e}^{-j\theta}\Gamma(j\omega)}\Gamma(j\omega)C_\mathrm{h}^{-1}(j\omega)\right|\\ &= \lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{Q^{-1}(j\omega) - \mathrm{e}^{j\theta}}{\mathrm{e}^{j\theta} - \Gamma(j\omega)}T(j\omega)C_\mathrm{h}^{-1}(j\omega)\right|\\ &\leq \left(\lim_{\epsilon\to 0}\sup_{\omega\in B_\epsilon(0)}\left|\frac{Q^{-1}(j\omega) - \Gamma(j\omega)}{\mathrm{e}^{j\theta} - \Gamma(j\omega)}\right| + 1\right)\left|T(0)C_\mathrm{h}^{-1}(0)\right|\end{aligned} \tag{31}$$

Since $\left|T(0)C_\mathrm{h}^{-1}(0)\right|$ is bounded, we will focus on the first term on the right hand side of (31):

$$\begin{aligned}&\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{Q^{-1}(j\omega) - \Gamma(j\omega)}{\mathrm{e}^{j\theta} - \Gamma(j\omega)}\right|\\ &\leq \lim_{\epsilon\to 0}\frac{\left|Q^{-1}(j\omega) - \Gamma(j\omega)\right|}{1 - |\Gamma(j\omega)|}\\ &= \lim_{\epsilon\to 0}\frac{\omega^2}{\sqrt{h^2\omega^2 + 1}\left|\tilde{L}(j\omega) - \omega^2\right| - \left|\tilde{L}(j\omega)\right|}\end{aligned} \tag{32}$$

Remember that $\tilde{L}(j\omega)$ is the open loop transfer function of a single subsystem without the two integrators. Around the origin we can therefore express $\tilde{L}(j\omega)$ as $\tilde{L}(j\omega) = a_0 + a_2\omega^2 + a_4\omega^4 + \ldots + j(a_1\omega + a_3\omega^3 + \ldots)$. Using that and L'Hôpital's Rule (32) becomes

$$\begin{aligned}
&\lim_{\epsilon\to 0} \sup_{(\omega,\theta)\in B_\epsilon(0,0)} \left| \frac{Q^{-1}(j\omega) - \Gamma(j\omega)}{e^{j\theta} - \Gamma(j\omega)} \right| \\
&\quad \leq \lim_{\omega\to 0} \left(\frac{1}{2} \frac{h^2}{\sqrt{h^2\omega^2+1}} \left| \tilde{L}(j\omega) - \omega^2 \right| \right. \\
&\quad\quad \left. + \sqrt{h^2\omega^2+1} \frac{\partial}{\partial\omega^2} \left| \tilde{L}(j\omega) - \omega^2 \right| - \frac{\partial}{\partial\omega^2} \left| \tilde{L}(j\omega) \right| \right)^{-1} \\
&\quad = \frac{1}{\frac{1}{2}h^2 a_0 - 1}. \qquad (33)
\end{aligned}$$

Thus using a time headway greater than $\sqrt{2/\tilde{L}(0)}$ will guarantee that $H_{\hat{e},\hat{d}}(s,z)$ is bounded at the NSSK.

To ensure $\left|H_{\hat{e},\hat{d}}\left(j\omega, e^{j\theta}\right)\right| < \infty$ for all $\omega \neq 0$ and θ, we must guarantee that $|\Gamma(j\omega)| < 1$ for all $\omega \neq 0$. Otherwise, since we know that $\Gamma(s)$ is strictly proper, there must exist an $\omega_0 \neq 0$ such that $|\Gamma(j\omega_0)| = 1$. To ensure $|\Gamma(j\omega)| < 1\ \forall\omega \neq 0$, the time headway h must be greater than the infimal time headway h_0:

$$h_0 := \sqrt{\sup_{\omega} \left(\frac{|T(j\omega)|^2 - 1}{\omega^2} \right)} \qquad (34)$$

where $T(s) = \frac{L(s)}{1+L(s)}$ is the single loop complementary sensitivity function of the system with zero time headway and $L(s) = \frac{1}{s^2}\tilde{L}(s)$ is the corresponding open loop transfer function with exactly two integrators and $\tilde{L}(0) \neq 0$.

Since the maximum in (34) can be attained at $\omega = 0$ or at at least one $\omega_0 \neq 0$, we will distinguish between these two cases:

(a) The maximum in (34) is attained at $\omega = 0$ only. Using L'Hôpital's Rule and the fact that $\tilde{L}(0) = \left|\tilde{L}(0)\right|$ condition (34) becomes

$$h_0 = \sqrt{2/\left|\tilde{L}(0)\right|}. \qquad (35)$$

(b) The maximum in (34) is attained at at least one $\omega_0 \neq 0$. In that case condition (34) becomes

$$h_0 = \sqrt{\left|\frac{L(j\omega_0)}{1+L(j\omega_0)}\right|^2 - 1} \Big/ \omega_0 \qquad (36)$$

Choosing a time headway which is strictly greater than h_0, $h > h_0$, will guarantee $|\Gamma(j\omega)| \leq 1$ for all ω and $|\Gamma(j\omega)| = 1$ only at $\omega = 0$ in both cases.

Note that if the supremum of (34) is achieved at $\omega = 0$ and h_0 is chosen according to (35) the essential supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega, e^{j\theta}\right)\right|$ is achieved at the origin. In that case approaching the origin along the curve $\theta = \theta(\omega) = -h\omega$ allows us to obtain the limit: $\lim_{\omega\to 0}\left|H_{\hat{e},\hat{d}}\left(j\omega, e^{\theta(\omega)}\right)\right| = \frac{1/2h^2a_0}{1/2h^2a_0-1}|C_h^{-1}(0)\Gamma(0)|$. This is in fact the supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega, e^{j\theta}\right)\right|$ over ω and θ obtained at the origin (compare with (31) and (33)).

However, if h_0 is chosen according to (36) the point where the essential supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega, e^{j\theta}\right)\right|$ is reached is continuous.

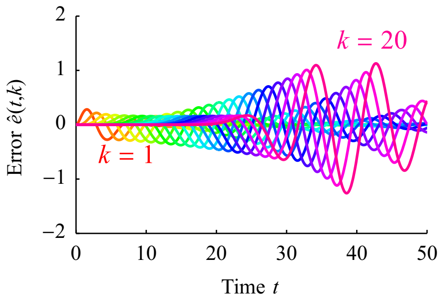

Fig. 1. Step Response for $h = 1$

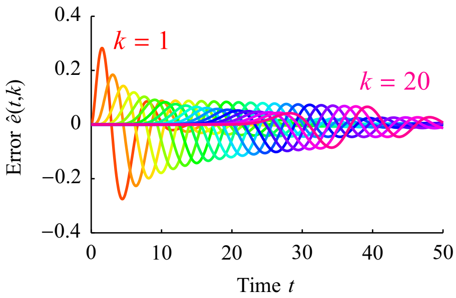

Fig. 2. Step Response for $h = 1.5$

C. Example and Simulations

To illustrate our results we will simulate a string of twenty vehicles with the open loop transfer function $L(s) = (s+1)/s^2$, such that the complementary sensitivity function for a single subsystem is $T(s) = (s+1)/(s^2+s+1)$. The infimal time headway is $h_0 \approx 1.47$, for details see [25, Fig. (a)]. Simulations for different time headways are shown in Figures 1 and 2. (Note that the step response is applied to the first vehicle in the string (drawn in red) and the disturbance travels trough the string towards the 20th vehicle (drawn in violet).)

V. Conclusions

We have analszed two-dimensional continuous-discrete systems using the combined Laplace-Z transform. We showed that the induced L_2 operator norm is equivalent to the L_∞ norm in the frequency domain. This was used to study the stability of a simple unidirectional homogeneous string of vehicles.

However, due to the special system architecture in a vehicle platoon a nonessential singularity of the second kind (NSSK) on the boundary of the stability region cannot be avoided. To show that the operator transfer function is also bounded at the NSSK requires a detailed discussion. Hence, the advantages of a standardised and short stability analysis using the induced operator norm in the frequency domain are reduced.

The authors will therefore work on more efficient possibilities to discuss the stability of two-dimensional continuous-discrete systems and platooning problems in particular.

VI. Appendix

To show that $\lim_{\epsilon\to 0}\int_{\mathcal{R}_\epsilon}|\hat{D}_\epsilon(j\omega,\mathrm{e}^{j\theta})|^2\mathrm{d}\omega\mathrm{d}\theta=(2\pi)^2$ we will start to evaluate the part of $\hat{D}_\epsilon$ depending on ω. Due to space constraints the rest will be summarised in

$$\hat{D}_{\epsilon_\theta}\left(\mathrm{e}^{j\theta}\right)=\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}. \tag{37}$$

(Note that ln is the natural logarithm.) Thus

$$\begin{aligned}&\iint\limits_{\omega,\theta\in\mathcal{R}_\epsilon}\frac{\left|\hat{D}_\epsilon\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2}{\left|\hat{D}_{\epsilon_\theta}\left(\mathrm{e}^{j\theta}\right)\right|^2}\mathrm{d}\omega\mathrm{d}\theta\\ &=\int\limits_{\theta\in\mathcal{R}_\epsilon}\alpha_{\omega_0}^2\frac{1}{8}\frac{1}{(\epsilon^2+\omega_0^2)\epsilon}\Bigg[-\omega_0\epsilon\ln\left(\omega_0^2+\epsilon^2+2\omega_0\omega+\omega^2\right)\\ &\quad+\left(4\epsilon^2+2\omega_0^2\right)\left(\arctan\left(\frac{\omega+\omega_0}{\epsilon}\right)+\arctan\left(\frac{\omega-\omega_0}{\epsilon}\right)\right)\\ &\quad+\omega_0\epsilon\ln\left(\omega_0^2+\epsilon^2-2\omega_0\omega+\omega^2\right)\Bigg]_{\partial\mathcal{R}_\epsilon}\mathrm{d}\theta.\end{aligned}$$

The next step is to evaluate the antiderivative for both peaks (around ω_0 and $-\omega_0$) and the limit for $\epsilon\to 0$. With (22) it yields

$$\lim_{\epsilon\to 0}\iint\limits_{\omega,\theta\in\mathcal{R}_\epsilon}\frac{\left|\hat{D}_\epsilon\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2}{\left|\hat{D}_{\epsilon_\theta}\left(\mathrm{e}^{j\theta}\right)\right|^2}\mathrm{d}\omega\mathrm{d}\theta=\int\limits_{\theta\in\mathcal{R}_\epsilon}2\pi\mathrm{d}\theta. \tag{38}$$

Also, the interval over $\hat{D}_{\epsilon_\theta}\left(\mathrm{e}^{j\theta}\right)$ has to be equal 2π. First, we will write $\left|\hat{D}_{\epsilon_\theta}\left(\mathrm{e}^{j\theta}\right)\right|^2$ as

$$\begin{aligned}|\hat{D}_{\epsilon_\theta}\left(\mathrm{e}^{j\theta}\right)|^2=&\alpha_{\theta_0}^2\Bigg(M+\frac{A}{1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{-j\theta}}+\frac{B}{1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{-j\theta}}\\ &+\frac{C}{1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{j\theta}}+\frac{D}{1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{j\theta}}\Bigg)\end{aligned} \tag{39}$$

with $A=D=\frac{1-\frac{\mathrm{e}^{-\epsilon}}{2}\left(\mathrm{e}^{j\theta_0}-\mathrm{e}^{-j\theta_0}\right)\left(\mathrm{e}^{-\epsilon+j\theta_0}-\mathrm{e}^{\epsilon-j\theta_0}\right)+\mathrm{e}^{2\epsilon}\cos^2\theta_0}{\left(1-\mathrm{e}^{-2\epsilon}\right)\left(1-\mathrm{e}^{-j2\theta_0}\right)\left(1-\mathrm{e}^{-2\epsilon+j2\theta_0}\right)}$, $B=C=\frac{1-\frac{\mathrm{e}^{-\epsilon}}{2}\left(\mathrm{e}^{j\theta_0}-\mathrm{e}^{-j\theta_0}\right)\left(\mathrm{e}^{\epsilon+j\theta_0}-\mathrm{e}^{-\epsilon-j\theta_0}\right)+\mathrm{e}^{2\epsilon}\cos^2\theta_0}{\left(1-\mathrm{e}^{-2\epsilon}\right)\left(1-\mathrm{e}^{j2\theta_0}\right)\left(1-\mathrm{e}^{-2\epsilon-j2\theta_0}\right)}$ and $M=\frac{\mathrm{e}^{-2\epsilon+4j\theta_0}-\mathrm{e}^{-2\epsilon+2j\theta_0}\cos^2\theta_0\left(1+\mathrm{e}^{-2\epsilon}\right)+\mathrm{e}^{-2\epsilon+2j\theta_0}-\mathrm{e}^{2j\theta_0}+\mathrm{e}^{-2\epsilon}}{\left(1-\mathrm{e}^{-2\epsilon}\right)\left(1-\mathrm{e}^{-2\epsilon+2j\theta_0}\right)\left(\mathrm{e}^{2j\theta_0}-\mathrm{e}^{-2\epsilon}\right)}$.

Evaluating the anti-derivative of $\left|\hat{D}_{\epsilon_\theta}\right|^2$ at $\mathcal{R}_\epsilon$ and allowing ϵ to go to 0 the integral can be simplified to

$$\begin{aligned}&\lim_{\epsilon\to 0}\int\limits_{\mathcal{R}_\epsilon}|\hat{D}_{\epsilon_\theta}\left(\mathrm{e}^{j\theta}\right)|^2\mathrm{d}\theta\\ &=\lim_{\epsilon\to 0}2\alpha_{\theta_0}^2(A+B)\left(\frac{\ln\left(1-\mathrm{e}^{-\epsilon}\mathrm{e}^{-j\sqrt{\epsilon}}\right)}{j}-\frac{\ln\left(1-\mathrm{e}^{-\epsilon}\mathrm{e}^{j\sqrt{\epsilon}}\right)}{j}\right)\\ &=\lim_{\epsilon\to 0}2\alpha_{\theta_0}^2(A+B)\left(\frac{\pi}{2}-\frac{-\pi}{2}\right)\end{aligned} \tag{40}$$

Using l'Hôspital's Rule we see that the limits of $\alpha_{\theta_0}^2A$ and $\alpha_{\theta_0}^2B$ for $\epsilon\to 0$ are 1/2. Therefore (40) becomes $\lim_{\epsilon\to 0}\int_{\mathcal{R}_\epsilon}\left|\hat{D}_{\epsilon_\theta}\left(\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\theta=2\pi$.

References

[1] W. S. Levine and M. Athans, "On the Optimal Error Regulation of a String of Moving Vehicles," *IEEE Transactions on Automatic Control*, vol. AC-11, no. 3, pp. 355–361, 1966.

[2] K.-C. Chu, "Decentralized Control of High-Speed Vehicular Strings," *Transportation Science*, vol. 8, no. 4, pp. 361–384, 1974.

[3] S. Sheikholeslam and C. A. Desoer, "Longitudinal control of a platoon of vehicles," in *Proceedings of the American Control Conference*, 1990, pp. 291–297.

[4] P. Barooah and J. P. Hespanha, "Error Amplification and Disturbance Propagation in Vehicle Strings with Decentralized Linear Control," in *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005*, 2005, pp. 4964–4969.

[5] P. Seiler, A. Pant, and J. K. Hedrick, "Disturbance Propagation in Vehicle Strings," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1835–1841, 2004.

[6] C. Chien and P. A. Ioannou, "Automatic Vehicle Following," in *Proc. American Control Conference*, 1992, pp. 1748–1752.

[7] S. Darbha and J. K. Hedrick, "String Stability of Interconnected Systems," *IEEE Trans. Autom. Contr.*, vol. 41, no. 3, pp. 349–357, 1996.

[8] J. Eyre, D. Yanakiev, and I. Kanellakopoulos, "A Simplified Framework for String Stability Analysis of Automated Vehicles," *Vehicle Systems Dynamic*, vol. 30, no. 5, pp. 375–405, November 1998.

[9] I. Lestas and G. Vinnicombe, "Scalability in Heterogeneous Vehicle Platoons," in *Proceedings of the American Control Conference*, 2007, pp. 4678–4683.

[10] J. Shanks, S. Treitel, and J. H. Justice, "Stability and Synthesis of Two-Dimensional Recursive Filters," *IEEE Trans. on Audio and Electroacoustics*, vol. AU-20, no. 2, pp. 115–128, 1972.

[11] T. S. Huang, "Stability of Two-Dimensional Recursive Filters," *IEEE Trans. Audio Electroacoustics*, vol. AU-20, no. 2, pp. 158–163, 1972.

[12] B. D. O. Anderson and E. I. Jury, "Stability Test for Two-Dimensional Recursive Filters," *IEEE Trans. on Audio and Electroacoustics*, vol. AU-21, no. 4, pp. 366–372, 1973.

[13] H. G. Ansell, "On certain two-variable generalizations of circuit theory, with applications to networks of transmission lines and lumped reactances," *IEEE Trans. on Circuit Theory*, vol. CT-11, no. 2, pp. 214–223, 1964.

[14] H. C. Reddy and P. K. Rajan, "A Simpler Test Set for Two-Variable Very Strict Hurwitz Polynomials," *Proc. of the IEEE*, vol. 74, no. 6, pp. 890–891, 1986.

[15] P. Agathoklis, E. I. Jury, and M. Mansour, "Algebraic Necessary and Sufficient Conditions for Very strict Hurwitz Property of a 2-D Polynomial," *Multidim. Systems and Signal Processing*, vol. 2, no. 1, pp. 45–53, 1991.

[16] C. Xiao, P. Agathoklis, and D. J. Hill, "On the Positive Definite Solutions to the 2-D Continuous-time Lyapunov Equation," *Multidim. Sys. and Signal Processing*, vol. 8, no. 3, pp. 315–333, 1997.

[17] D. Goodman, "Some Stability Properties of Two-Dimensional Linear Shift-Invariant Digital Filters," *IEEE Trans. on Circuits and Systems*, vol. CAS-24, no. 4, pp. 201–208, 1977.

[18] S. A. Dautov, "On Absolute Convergence of the Series of Taylor Coefficients of a Rational Function of Two-Variables. Stability of Two-Dimensional Digital Filters," *Soviet Mathematics Doklady*, vol. 23, no. 2, pp. 448–451, 1981.

[19] H. C. Reddy and E. I. Jury, "Study of the BIBO Stability of 2-D Recursive Digital Filters in the Presence of Nonessential Singularties of the Second kind – Analog Approach," *IEEE Trans. on Circuits and Systems*, vol. CAS-34, no. 3, pp. 280–284, 1987.

[20] Y. Xiao and M. H. Lee, "2-D Laplace-Z Transformation," *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, vol. E89-A, no. 5, pp. 1500–1504, May 2006.

[21] R. E. Bellmann and R. S. Roth, *The Laplace Transform*. World Scientific, 1984.

[22] J. H. Mathews and R. W. Howell, *Complex Analysis for Mathematics and Engineering*. Jones & Bartlett Publishers, 2006.

[23] L. Debnath and D. Bhatta, *Integral Transforms and their Applications*. CRC Press, 2006.

[24] T. Y. Lee, *Henstock-Kurzweil Integration on Euclidean Spaces (Series in Real Analysis)*. World Scientific Publishing Company, 2011.

[25] S. Klinge and R. H. Middleton, "Time Headway Requirements for String Stability of Homogeneous Linear Unidirectionally Connected Systems," in *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, 2009, pp. 1992–1997.