Stability of Two-Dimensional Linear Systems With Singularities on the Stability Boundary Using LMIs

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***Abstract*—This paper gives results on stability and asymptotic stability of two-dimensional systems using linear matrix inequalities (LMIs). Despite a long history of research in this area, systems with singularities on the stability boundary (SSB) have received limited attention because they cannot produce a sign definite solution to the required LMI. However, 2D systems describing some classes of models of vehicle platoons generically involve an SSB. Therefore, commonly used definitions for (asymptotic) stability and strict LMI conditions are not suitable to discuss the stability of these systems. It is shown that the existence of a negative *semi*definite solution together with simple additional conditions is sufficient to guarantee asymptotic stability. Thus, the stability conditions discussed here can be used to study a wider range of dynamical systems, including systems with singularities on the stability boundary (SSB), which cannot be exponentially stable. A unified framework is used to analyse continuous-continuous, continuous-discrete and discrete-discrete systems simultaneously.**

***Index Terms*—Linear matrix inequalities (LMIs), stability analysis, two-dimensional (2-D) systems**

I. Introduction

In this paper stability of two-dimensional (2D) linear systems will be examined. 2D refers to the fact that signals and variables depend on two independent variables. Since both variables can be continuous or discrete, most analyses distinguish between discrete, continuous and continuous-discrete 2D systems. The majority of the past research focuses on discrete 2D systems due to the range of applications for this case.

Early stability results on discrete 2D systems used 2D Z transforms, involving functions of two complex variables, z_1, z_2. [1] analysed the input-output system $Y(z_1,z_2) = \text{num}(z_1,z_2)/\text{den}(z_1,z_2) \cdot W(z_1,z_2)$ and claimed[1] it is BIBO stable if and only if the characteristic polynomial, $\text{den}(z_1,z_2)$, has no zeros in the closed unit bi-disc $\overline{U}^2 = \{(z_1,z_2) : |z_1| \leq 1, |z_2| \leq 1\}$. This led to different stability tests such as [2] and [3], or for the continuous time case [4]. Necessary and sufficient conditions to guarantee a given polynomial is 'very strictly Hurwitz' (for the continuous case) were published in (for example) [5] and [6].

A. Singularities on the Stability Boundary

An important special case is often neglected in discussions of 2D systems stability. This is the case when there exists a Singularity on the Stability Boundary (SSB). In discrete time, this means that there exists a set of (z_1,z_2), such that $|z_1| = |z_2| = 1$ and $\text{den}(z_1,z_2) = 0$. In a 2D transfer function setting, and where the numerator is simultaneously zero ($\text{num}(z_1,z_2) = 0$), this is referred to as a Non-essential Singularity of the Second Kind (NSSK) on the stability boundary.

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[1]The claim is correct except for some cases with singularities on the stability boundary.

Note that it may not be straightforward to detect if an NSSK is present. Also, it may be tempting to treat such cases as esoteric and irrelevant in practice. However, the existence of an NSSK cannot always be avoided and might even be desirable or a structural requirement. For example, the design of fan filters inherently require an NSSK on the stability boundary, [7]. Also, as we will show later, some classes of models of vehicle platoons generically involve an SSB.

It was shown in [8] that transfer functions with NSSK on the stability boundary may be either BIBO stable or unstable. Thus, the location of the poles of a 2D transfer function alone does not determine BIBO stability of the system. It was shown in [9] that the system is stable if the transfer function has finitely many NSSK on the stability boundary and can be continuously extended to the closed polydisc. [10] gave a necessary condition for stability, namely that there should be no NSSK inside the open unit bi-disk.

Thus we see that the analysis of stability of marginally stable 2D systems (that is those with an SSB) is a subtle issue that needs careful examination. We next turn to discuss closely related stability questions from the perspective of internal stability.

B. Internal Stability and 2D State Space Systems

Different state space models have been presented to describe discrete 2D systems in the time domain. Two well known models are Fornasini-Marchesini's first model (FM1), [11], and second model (FM2), [12]. Although FM2 has attracted the most attention, a necessary condition for asymptotic stability for FM1 appeared in [13]. Sufficient LMI conditions for asymptotic stability were developed in [14] and necessary and sufficient conditions in [15].

In [12] the authors proved asymptotic stability for FM2. A straight diagonal separation set or "contour" and its norm $\|\mathcal{X}_r\| = \sup_{n\in\mathbb{Z}} |\boldsymbol{x}(r-n,n)|$ are defined. In this line of work, asymptotic stability is defined as $\|\mathcal{X}_0\| < \infty$ implies $\lim_{r\to\infty} \|\mathcal{X}_r\| = 0$. This is true if and only if the characteristic polynomial is non-zero for any (z_1,z_2) in $\overline{U}^2$. Note that this, as with the other definitions of asymptotic stability, also implies 2D exponential stability. An extension can be found in [16] where the authors prove asymptotic stability using a more general contour. The result uses a linear matrix inequality constraint requiring a positive definite Hermitian solution $\boldsymbol{P}(\omega)$ for all real ω.

Based on the necessary and sufficient condition on the characteristic polynomial to be devoid of zeros in $\overline{U}^2$, a sufficient LMI based condition for asymptotic stability was derived in [17] providing the first LMI condition for FM2 with constant coefficients. Necessary and sufficient conditions with constant coefficients for asymptotic stability were presented in [18].

Note that the definition of asymptotic stability used in the references on FM2 above implies that for any set of L_∞ bounded boundary conditions, the states tend to zero as both independent variables tend to infinity.

Another widely used state space description was presented by

Roesser in [19]:

$$\begin{pmatrix} \boldsymbol{x}_1(k+1,l) \\ \boldsymbol{x}_2(k,l+1) \end{pmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix}}_{\boldsymbol{A}} \begin{pmatrix} \boldsymbol{x}_1(k,l) \\ \boldsymbol{x}_2(k,l) \end{pmatrix} + \underbrace{\begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{pmatrix}}_{\boldsymbol{B}} \boldsymbol{u}(k,l) \tag{1}$$

$$\boldsymbol{y}(k,l) = \underbrace{\begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix}}_{\boldsymbol{C}} \begin{pmatrix} \boldsymbol{x}_1(k,l) \\ \boldsymbol{x}_2(k,l) \end{pmatrix} + \boldsymbol{D}\boldsymbol{u}(k,l) \tag{2}$$

where $\boldsymbol{x}_1(k,l) \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2(k,l) \in \mathbb{R}^{n_2}$ and the dimensions of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are chosen appropriately.

It was claimed in [20] that the characteristic polynomial $\mathrm{den}(z_1,z_2)$ fulfils Shanks' stability criterion [1] if and only if there exists a positive definite, symmetric matrix $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$, where $\oplus$ denotes the direct sum, i. e. $\boldsymbol{P}_1 \oplus \boldsymbol{P}_2 = \mathrm{diag}\{\boldsymbol{P}_1,\boldsymbol{P}_2\}$, $\boldsymbol{P}_1 \in \mathbb{R}^{n_1 \times n_1}$ and $\boldsymbol{P}_2 \in \mathbb{R}^{n_2 \times n_2}$, such that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} = \boldsymbol{Q} < 0$. However, Anderson *et al.* later showed that for discrete 2D systems, in general, the existence of such a $\boldsymbol{P}$ is sufficient but not necessary for stability, [21].

A continuous analogue of the Roesser model is widely used to describe linear continuous 2D systems. It was claimed in [22] that a continuous 2D system is stable (characteristic polynomial is very strictly Hurwitz) if and only if there exists a positive definite, symmetric matrix $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$ such that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = \boldsymbol{Q} < 0$. Again however, the existence of such a $\boldsymbol{P}$ is sufficient but not necessary for stability [21]. For continuous systems, necessary and sufficient LMI conditions for the existence of such a $\boldsymbol{P}$ appeared in [23].

Another necessary and sufficient LMI condition to ensure that the characteristic polynomial is very strict Hurwitz, that requires the existence of a positive definite Hermitian solution $\boldsymbol{P}(\omega)$ for all real ω was given in [6]. Necessary and sufficient LMI conditions for asymptotic stability with constant matrices are presented in [24]. Piekarski's LMI condition $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = \boldsymbol{Q} < 0$ was later found to be sufficient to guarantee asymptotic stability for continuous systems with bounded initial conditions, [25].

Alongside discrete repetitive processes (modelled as discrete 2D systems) some researchers also studied "differential" repetitive processes leading to the study of continuous-discrete 2D systems. Stability theory for continuous-discrete 2D systems appears to be well developed. Different conditions for stability and asymptotic stability of differential repetitive processes with dynamic boundary conditions (depending on the pass profiles of the previous passes) are given in [26]. These results were extended to stability tests based on a one-dimensional Lyapunov function in [27].

In [28] the authors discuss stability along the pass (similar to asymptotic stability) for differential repetitive processes modelled in a form similar to the Roesser model. They claim that such a system is stable along the pass if there exist two positive definite, symmetric matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{P}_1 \oplus \boldsymbol{0}) + (\boldsymbol{P}_1 \oplus \boldsymbol{0})\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{0} \oplus \boldsymbol{P}_2)\boldsymbol{A} - (\boldsymbol{0} \oplus \boldsymbol{P}_2) = \boldsymbol{Q} < 0$. The proof in [28] refers to [29] for details. While this book covers extensive results in the area, a complete LMI based stability proof for the Roesser Model is not given.

It should also be noted here that similar to the results on the Fornasini-Marchesini models the commonly used definitions of asymptotic stability for the Roesser model require that for any set of L_∞ bounded initial or boundary conditions, the states tend to 0 as $t_1,t_2 \to \infty$.

C. A Motivating Example

To motivate consideration of 2D systems with an SSB, we give an example of a class of problems where an SSB is generic. Consider the situation of vehicle platooning. In order to achieve tight spacing between vehicles travelling in a string (or "platoon"), suppose the vehicles have an automatic controller for longitudinal position. This controller uses local measurements to regulate the distance to the predecessor, or in the case of the lead vehicle, to follow a given trajectory.

Assume that the local state space variables of the kth vehicle (such as its position $\hat{x}(t,k)$, velocity $\hat{v}(t,k)$ and controller states) are summarised in the vector $\boldsymbol{x}_1(t,k) \in \mathbb{R}^{n_1}$. Further, assume the position of the preceding vehicle $\hat{x}(t,k-1)$, that is used as a reference for the kth vehicle, is set to be the scalar $\hat{x}(t,k-1) = x_2(t,k) \in \mathbb{R}$. The overall 2D system can be described by

$$\begin{aligned} \begin{pmatrix} \dot{\boldsymbol{x}}_1(t,k) \\ \Delta x_2(t,k) \end{pmatrix} &= \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_1(t,k) \\ x_2(t,k+1) - x_2(t,k) \end{pmatrix} \\ &= \underbrace{\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & -1 \end{bmatrix}}_{=:\boldsymbol{A}} \begin{pmatrix} \boldsymbol{x}_1(t,k) \\ x_2(t,k) \end{pmatrix}. \end{aligned} \tag{3}$$

For further details on this model, refer to the examples in Section V. The characteristic polynomial for the 2D system in (3) is:

$$p_{ch}(s,z) := \det\left(\begin{bmatrix} s\mathbf{I} & 0 \\ 0 & (z-1)\mathbf{I} \end{bmatrix} - \boldsymbol{A} \right). \tag{4}$$

Now suppose, as we normally require for platoon following, that in response to a unit step, $x_2(t,0) = 1$, that a steady state is reached where all vehicles follow an identical reference with zero steady state error. The condition of identical steady state references means $\lim_{t\to\infty} x_2(t,k) = 1, \forall k$. Furthermore, from (3), for this to be an equilibrium, there must exist an $\overline{x}_1$ such that $\boldsymbol{A}_{11}\overline{x}_1 + \boldsymbol{A}_{12} = 0$ and $\boldsymbol{A}_{21}\overline{x}_1 = 1$. This immediately implies that $\boldsymbol{A}$ is singular. Since $\boldsymbol{A}$ is singular, (4) implies immediately that we have an SSB, since $p_{ch}(0,1) = det(\boldsymbol{A}) = 0$.

Hence, the presence of this SSB is a consequence of the structure of the vehicle platoon system. It can also be shown that a 2D system describing a vehicle platoon exhibits an SSB at $s = 0$ and $z = 1$ for more general settings with $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$ (see [30] for further details).

Note that in a similar approach, [31, Example 2], a single lane of a motorway is modelled as a 2D positive system. The authors divide the vehicles into groups depending on their speed and investigate the number of vehicles of each group per time interval and stretch of the highway. Here, however, we are interested in studying the behaviour of individual vehicles whose control actions explicitly depend on the behaviour of the preceding vehicles.

D. Contribution

Previous LMI based results in the time domain exclude systems where $\boldsymbol{A}$ has an SSB since a sign definite solution of the LMI is required. However, as we will show later in Lemma 1, such systems never admit a sign definite solution to the required LMI. (Indeed it will be shown in Lemma 2 that if the system exhibits an SSB at $s = 0$ and $z = 1$ there exists no Lyapunov function with a negative definite divergence.) Hence, all LMI based stability conditions presented in the literature so far cannot be employed to study the stability of 2D systems including an SSB. In particular, none of the previous LMI based results are suitable for studying vehicle platoon string stability.

This issue is closely related to the definitions used for asymptotic stability. Most common definitions for asymptotic stability require the states to tend to zero in the presence of any set of L_∞ bounded initial conditions. As discussed above in Section I-C it is highly desirable that applying a bounded initial condition (i.e. a step signal as a reference) in a vehicle platoon leads to convergence of all states to a nonzero equilibrium. Therefore, these previous definitions of asymptotic stability are uninformative in this context.

In this paper we fill this gap. We extend previous 2D LMI based stability results to the case of systems with SSB(s). We use an alternate notion of asymptotic stability that proves useful in the context of our motivating example. We give results that permit asymptotic stability proofs for some systems with SSBs using a 2D Lyapunov function with negative semi-definite divergence.

Our paper is structured as follows. After clarifying the notation in Section II we will discuss preliminary results in Section III (including stability of 2D systems in Corollary 1). Our main result on asymptotic stability is presented in Section IV. Illustrative results are given in Section V before concluding remarks and suggestions for further work in Section VI.

II. Notation

We will study stability of 2D systems using a unified notation to describe the stability of the state variable $\boldsymbol{x}(t_1,t_2)$ where for $i \in \{1,2\}$

$$t_i \in \mathbb{T}_i \quad \text{that is} \quad t_i \in \begin{cases} \mathbb{R}^+ : & t_i \text{ continuous,} \\ \mathbb{N} : & t_i \text{ discrete.} \end{cases} \tag{5}$$

We will use the generalised derivative operator $\delta_i; i \in \{1,2\}$ to represent either a derivative (continuous) or forward difference (discrete) with respect to t_i. For example:

$$\delta_1 \boldsymbol{x}(t_1,t_2) := \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t_1} \boldsymbol{x}(t_1,t_2) : & t_1 \text{ continuous,} \\ \boldsymbol{x}(t_1+1,t_2) - \boldsymbol{x}(t_1,t_2) : & t_1 \text{ discrete.} \end{cases} \tag{6}$$

The generalised integration operator $\mathcal{S}$ is defined as regular integration in continuous time, or left Riemann summation in discrete time. For example:

$$\overset{b}{\underset{a}{\mathcal{S}}} \boldsymbol{x}(t_1,t_2)\mathrm{d}t_1 := \begin{cases} \int_a^b \boldsymbol{x}(t_1,t_2)\mathrm{d}t_1 : & t_1 \text{ continuous,} \\ \sum_{t_1=a}^{t_1=b-1} \boldsymbol{x}(t_1,t_2) : & t_1 \text{ discrete.} \end{cases} \tag{7}$$

We will consider autonomous 2D systems of the following form (Roesser model, [19])

$$\underbrace{\begin{pmatrix} \delta_1 \boldsymbol{x}_1(t_1,t_2) \\ \delta_2 \boldsymbol{x}_2(t_1,t_2) \end{pmatrix}}_{\delta \boldsymbol{x}(t_1,t_2)} = \underbrace{\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} \boldsymbol{x}_1(t_1,t_2) \\ \boldsymbol{x}_2(t_1,t_2) \end{pmatrix}}_{\boldsymbol{x}(t_1,t_2)} \tag{8}$$

where $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$, with the initial conditions $\boldsymbol{x}_1(0,t_2) = \boldsymbol{x}_{10}(t_2)$ and $\boldsymbol{x}_2(t_1,0) = \boldsymbol{x}_{20}(t_1)$. The autonomous system (8) has a solution that satisfies:

$$\boldsymbol{x}_1(t_1,t_2) = \mathrm{E}(\boldsymbol{A}_{11})^{t_1} \boldsymbol{x}_{10}(t_2) + \overset{t_1}{\underset{0}{\mathcal{S}}} \mathrm{E}(\boldsymbol{A}_{11})^{\tau} \boldsymbol{A}_{12} \boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2)\mathrm{d}\tau, \tag{9}$$

$$\boldsymbol{x}_2(t_1,t_2) = \mathrm{E}(\boldsymbol{A}_{22})^{t_2} \boldsymbol{x}_{20}(t_1) + \overset{t_2}{\underset{0}{\mathcal{S}}} \mathrm{E}(\boldsymbol{A}_{22})^{\tau} \boldsymbol{A}_{21} \boldsymbol{x}_1(t_1, t_2 - \mathbb{I}_2 - \tau)\mathrm{d}\tau, \tag{10}$$

where $\mathbb{I}_i$ for $i \in \{1,2\}$ denotes the indicator function

$$\mathbb{I}_i := \begin{cases} 0 : & t_i \text{ continuous,} \\ 1 : & t_i \text{ discrete,} \end{cases} \tag{11}$$

and the generalised exponential

$$\mathrm{E}(\boldsymbol{A})^t := \begin{cases} \mathrm{e}^{\boldsymbol{A}t} : & t \text{ continuous,} \\ (\mathbf{I} + \boldsymbol{A})^t : & t \text{ discrete.} \end{cases} \tag{12}$$

We say $\boldsymbol{A}$ is stable to mean either $\boldsymbol{A}$ is Hurwitz stable (continuous case) or $\mathbf{I} + \boldsymbol{A}$ is Schur stable (discrete case). In either case, if $\boldsymbol{A}$ is stable, then there exist $\lambda > 0$ (and in addition $\lambda < 1$ in the discrete case) and $k < \infty$ such that

$$\left\|\mathrm{E}(\boldsymbol{A})^t\right\| \leq k\mathrm{E}(-\lambda)^t. \tag{13}$$

Note that

$$\overset{t}{\underset{0}{\mathcal{S}}} \mathrm{E}(-\lambda)^{\tau}\mathrm{d}\tau = \frac{1 - \mathrm{E}(-\lambda)^t}{\lambda} \tag{14}$$

and for $-\lambda$ stable

$$\overset{\infty}{\underset{t}{\mathcal{S}}} \mathrm{E}(-\lambda)^{\tau}\mathrm{d}\tau = \frac{\mathrm{E}(-\lambda)^t}{\lambda}. \tag{15}$$

Moreover $\oplus$ denotes the direct sum of matrices, e. g. $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2 = \mathrm{diag}\{\boldsymbol{P}_1, \boldsymbol{P}_2\}$, $\mathbf{I}$ and $\mathbf{0}$ denote the identity matrix and the zero matrix, respectively, of appropriate dimensions and the imaginary unit is denoted by j. Consider the 2D vector Lyapunov function

$$\boldsymbol{V}(t_1,t_2) := \begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}}(t_1,t_2) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{x}_2^{\mathrm{T}}(t_1,t_2) \end{bmatrix} \boldsymbol{P}\boldsymbol{x}(t_1,t_2) = \begin{pmatrix} V_1(t_1,t_2) \\ V_2(t_1,t_2) \end{pmatrix} \tag{16}$$

with $\boldsymbol{P}_1 = \boldsymbol{P}_1^{\mathrm{T}} > 0$, $\boldsymbol{P}_2 = \boldsymbol{P}_2^{\mathrm{T}} > 0$, $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$ and

$$\mathrm{div}\, \boldsymbol{V}(t_1,t_2) = \delta_1 V_1(t_1,t_2) + \delta_2 V_2(t_1,t_2) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} \tag{17}$$

with $\delta_i V_i(t_1,t_2) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}_i\boldsymbol{x}$ and

$$\boldsymbol{Q} = \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \quad \text{where} \quad \boldsymbol{Q}_i = \boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_i + \tilde{\boldsymbol{P}}_i\boldsymbol{A} + \mathbb{I}_i\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_i\boldsymbol{A} \tag{18}$$

for $i \in \{1,2\}$ with $\tilde{\boldsymbol{P}}_1 = (\boldsymbol{P}_1 \oplus \mathbf{0})$ and $\tilde{\boldsymbol{P}}_2 = (\mathbf{0} \oplus \boldsymbol{P}_2)$.

Further ξ_i for $i \in \{1,2\}$ is the Laplace variable s_i if t_i is continuous or the Z transform variable z_i if t_i is discrete.

Definition 1 (Singularity on the Stability Boundary (SSB))**.** The 2D Roesser Model has a singularity on the stability boundary if there exits a set of ω_i (t_i continuous) or θ_i (t_i discrete) such that the matrix $((\xi_1 - \mathbb{I}_1)\mathbf{I} \oplus (\xi_2 - \mathbb{I}_2)\mathbf{I}) - \boldsymbol{A}$ is singular for $\xi_i = j\omega_i$ or $\xi_i = \mathrm{e}^{j\theta_i}$, respectively.

We will make use of the following different definitions of initial conditions.

Definition 2 (L_2 and L_∞ Bounded Initial Conditions)**.** We say the initial conditions of a 2D Roesser Model are "Bounded" if they are in L_2 and L_∞, that is there exist $c_i, \zeta_i < \infty$ such that for $i \in \{1,2\}$

$$\|\boldsymbol{x}_{i0}(\cdot)\|_2^2 = \overset{\infty}{\underset{0}{\mathcal{S}}} |\boldsymbol{x}_{i0}(t)|^2\mathrm{d}t \leq c_i, \quad \text{and} \tag{19}$$

$$\|\boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t \geq 0} |\boldsymbol{x}_{i0}(t)| \leq \zeta_i. \tag{20}$$

Definition 3 (L_2' and L_∞'' Smooth Bounded Initial Conditions)**.** We say the initial conditions of a 2D Roesser Model are Smooth Bounded Initial Conditions if they are L_2 and L_∞ bounded according to Definition 2, and in addition there exist $c_i', \zeta_i', \zeta_i'' < \infty$ such that for $i \in \{1,2\}$

$$\|\delta\boldsymbol{x}_{i0}(\cdot)\|_2^2 = \overset{\infty}{\underset{0}{\mathcal{S}}} |\delta\boldsymbol{x}_{i0}(t)|^2\mathrm{d}t \leq c_i', \tag{21}$$

$$\|\delta\boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t>0} |\delta\boldsymbol{x}_{i0}(t)| \leq \zeta_i', \quad \text{and} \tag{22}$$

$$\|\delta^2\boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t>0} |\delta^2\boldsymbol{x}_{i0}(t)| \leq \zeta_i''. \tag{23}$$

We will discuss the stability of 2D systems according to the following definitions.

Definition 4 (Stability of 2D Roesser Model)**.** The autonomous 2D Roesser Model (8) is stable if for each $M > 0$ there exists a set of $c_i(M), \zeta_i(M) > 0$ such that if the initial conditions are in L_2 and L_∞ with bounds c_i and ζ_i for $i \in \{1,2\}$, respectively, then

$$|\boldsymbol{x}(t_1,t_2)| \leq M \quad \text{for all } t_1,t_2 > 0. \tag{24}$$

Definition 5 (Asymptotic Stability of 2D Roesser Model with Smooth Bounded Initial Conditions)**.** The autonomous two-dimensional Roesser Model (8) is asymptotically stable, if for any Smooth Bounded Initial Conditions (according to Definition 3) it is stable, and the following limit holds for $i \in \{1,2\}$

$$\lim_{t_1+t_2\to\infty} \boldsymbol{x}_i(t_1,t_2) = 0. \tag{25}$$

Note that asymptotic stability requires the states to tend to zero as $t_1 + t_2 \to \infty$. That includes the cases where $t_1 \to \infty$, $t_2 \to \infty$ and the double limit $\lim_{t_1,t_2\to\infty}$ where t_1 and t_2 tend to $+\infty$ at the same time but in any possible form and direction.

III. Mathematical Preliminaries

Before presenting our results concerning the asymptotic stability of 2D systems we would like to show the connection between singularities on the stability boundary and the Lyapunov function.

Lemma 1. *Consider the autonomous 2D system (8). If the system has a singularity on the stability boundary (SSB), then for every symmetric choice of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, there exists a vector $\boldsymbol{v}$ such that $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{v} = 0$ where $\boldsymbol{Q}$ is given in (18).*

Proof: The characteristic polynomial is equal to $\det(((\xi_1 - \mathbb{I}_1)\mathbf{I} \oplus (\xi_2 - \mathbb{I}_2)\mathbf{I}) - \boldsymbol{A})$. Since the system has a singularity at $\xi_i = j\omega_i$ or $\xi_i = \mathrm{e}^{j\theta_i}$, respectively, the matrix $((\xi_1 - \mathbb{I}_1)\mathbf{I} \oplus (\xi_2 - \mathbb{I}_2)\mathbf{I}) - \boldsymbol{A}$ is singular for $\xi_i = j\omega_i$ or $\xi_i = \mathrm{e}^{j\theta_i}$, respectively. Therefore, there exists a non-zero vector $\boldsymbol{v} \in \mathbb{C}^n$ such that

$$\left(\begin{bmatrix}(\xi_1 - \mathbb{I}_1)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (\xi_2 - \mathbb{I}_2)\mathbf{I}\end{bmatrix} - \boldsymbol{A}\right)\boldsymbol{v} = \mathbf{0}. \tag{26}$$

Using (26) we can rewrite $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} = \boldsymbol{v}^{\mathrm{H}}(\boldsymbol{Q}_1 + \boldsymbol{Q}_2)\boldsymbol{v}$ and see from (18) that for instance if t_1 is continuous and t_2 is discrete

$$\begin{aligned}\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} &= \boldsymbol{v}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_1\boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}}\tilde{\boldsymbol{P}}_1\boldsymbol{A} \pm \boldsymbol{v}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_2\boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}}\tilde{\boldsymbol{P}}_2\boldsymbol{A}\boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_2\boldsymbol{A}\boldsymbol{v} \\ &= \boldsymbol{v}^{\mathrm{H}}\begin{bmatrix}(-j\omega_1 + j\omega_1)\,\boldsymbol{P}_1 & \mathbf{0} \\ \mathbf{0} & \left(\mathrm{e}^{-j\theta_2} - 1 + \mathrm{e}^{j\theta_2} - 1 + 1 - \mathrm{e}^{-j\theta_2} - \mathrm{e}^{j\theta_2} + 1\right)\boldsymbol{P}_2\end{bmatrix}\boldsymbol{v} \\ &= 0.\end{aligned} \tag{27}$$

If t_1 is discrete or t_2 is continuous it can be shown in a similar way that $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} = 0$. Thus, $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} = 0$ independently of $\boldsymbol{P}$. ■

Note therefore that, for a system including SSB it is not possible to find positive definite matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that $\boldsymbol{Q}$ is sign definite. Thus, there exists no quadratic 2D Lyapunov function with negative definite divergence. Assuming that the system exhibits a SSB at $s = 0$ and $z = 1$, we can further show that there exists no Lyapunov function of any form with a negative definite divergence.

Lemma 2. *Consider the autonomous 2D system (8). If the system has a singularity on the stability boundary (SSB) at $s_i = j\omega_i = 0$ (in case t_i is continuous) and $z_i = \mathrm{e}^{j\theta_i} = 1$ (i.e. $\theta_i = 0$ in case t_i is discrete), then for every choice of a 2D Lyapunov function $\boldsymbol{V} = \begin{pmatrix}V_1(\boldsymbol{x}_1) & V_2(\boldsymbol{x}_2)\end{pmatrix}^{\mathrm{T}}$ there exists a set $\overline{\boldsymbol{x}}_1 \neq 0$ and $\overline{\boldsymbol{x}}_2 \neq 0$ such that for all scalar $\gamma \neq 0$: $\delta_1 V_1(\boldsymbol{x}_1)|_{(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)} = 0$ and $\delta_2 V_2(\boldsymbol{x}_2)|_{(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)} = 0$ and thus $\operatorname{div}\boldsymbol{V}|_{(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)} = 0$.*

Proof: Since the system has a singularity at $\xi_i = j\omega_i = 0$ or $\xi_i = \mathrm{e}^{j\theta_i} = 1$, respectively, there exists a non-zero vector $\boldsymbol{v} \in \mathbb{C}^n$ such that (26) is satisfied for $\xi_i = j\omega_i = 0$ or $\xi_i = \mathrm{e}^{j\theta_i} = 1$, respectively, and thus $\boldsymbol{A}\boldsymbol{v} = \mathbf{0}$ and thus $\gamma\boldsymbol{A}\boldsymbol{v} = \mathbf{0}$. Choosing $\overline{\boldsymbol{x}}_1$ and $\overline{\boldsymbol{x}}_2$ such that $\boldsymbol{v} = \begin{pmatrix}\overline{\boldsymbol{x}}_1 \\ \overline{\boldsymbol{x}}_2\end{pmatrix}$ yields $\delta\boldsymbol{x}|_{(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)} = \gamma\boldsymbol{A}\boldsymbol{v} = \mathbf{0}$.

If t_1 is continuous, $\delta_1 V_1(\boldsymbol{x}_1)|_{(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)} = \left(\frac{\mathrm{d}V_1}{\mathrm{d}\boldsymbol{x}_1}\right)^{\mathrm{T}} \frac{\mathrm{d}\boldsymbol{x}_1}{\mathrm{d}t_1}\Big|_{(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)} = \left(\frac{\mathrm{d}V_1}{\mathrm{d}\boldsymbol{x}_1}\right)^{\mathrm{T}} \delta_1\boldsymbol{x}_1\Big|_{(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)} = 0$. If t_1 is discrete note that $\delta_1 V_1(\boldsymbol{x}_1) = \Delta_1 V_1(\boldsymbol{x}_1) = V_1(\boldsymbol{x}_1(t_1+1,t_2)) - V_1(\boldsymbol{x}_1(t_1,t_2))$. Setting $(\boldsymbol{x}_1,\boldsymbol{x}_2) = (\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)$ implies $\boldsymbol{x}_1(t_1+1,t_2) = \boldsymbol{x}_1(t_1,t_2) = \gamma\overline{\boldsymbol{x}}_1$ and thus $\delta_1 V_1(\boldsymbol{x}_1)|_{(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\gamma\overline{\boldsymbol{x}}_1,\gamma\overline{\boldsymbol{x}}_2)} = V_1(\gamma\overline{\boldsymbol{x}}_1) - V_1(\gamma\overline{\boldsymbol{x}}_1) = 0$.

Hence, for systems that exhibit a SSB at $s_i = 0$ or $z_i = 1$, respectively, there exists no 2D Lyapunov function with negative definite divergence. ■

Even though for systems including SSB $\boldsymbol{Q}$ can never be sign definite, the existence of a negative semi-definite $\boldsymbol{Q}$ together with some additional assumptions on $\boldsymbol{A}$ might be sufficient for stability. Furthermore, with some additional assumptions on the initial conditions we are able to guarantee asymptotic stability (with bounded smooth initial conditions).

Before we show stability we will first use some interesting properties of 2D non-negative vector fields with non-positive divergence.

Lemma 3. *Consider the 2D space of two variables t_1 and t_2 and the 2D non-negative vector field $\boldsymbol{V}^{\mathrm{T}}(t_1,t_2) = (V_1(t_1,t_2), V_2(t_1,t_2))$. If the divergence of the vector field $\boldsymbol{V}(t_1,t_2)$ is non-positive for every t_1 and t_2, then the generalised integral of $V_1(t_1,t_2)$ and $V_2(t_1,t_2)$ over $t_2 \in [0,T_2]$ and $t_1 \in [0,T_1]$, respectively, is bounded by the initial conditions $V_1(0,t_2)$ and $V_2(t_1,0)$, that is for all $T_1,T_2 > 0$:*

$$\mathop{S}_0^{T_2} V_1(T_1,t_2)\mathrm{d}t_2 \le \mathop{S}_0^{T_2} V_1(0,t_2)\mathrm{d}t_2 + \mathop{S}_0^{T_1} V_2(t_1,0)\mathrm{d}t_1 \tag{28}$$

$$\mathop{S}_0^{T_1} V_2(t_1,T_2)\mathrm{d}t_1 \le \mathop{S}_0^{T_2} V_1(0,t_2)\mathrm{d}t_2 + \mathop{S}_0^{T_1} V_2(t_1,0)\mathrm{d}t_1. \tag{29}$$

Proof: To prove this lemma we will simply consider the generalised surface integral of the divergence of $\boldsymbol{V}(t_1,t_2)$ over the rectangular region $t_1 \in [0,T_1]$, $t_2 \in [0,T_2]$:

$$W(T_1,T_2) := \mathop{S}_0^{T_2}\mathop{S}_0^{T_1} \left(\delta_1 V_1(t_1,t_2) + \delta_2 V_2(t_1,t_2)\right)\mathrm{d}t_1\mathrm{d}t_2 \tag{30}$$

Using the fundamental theorem of calculus or Gauss Divergence Theorem for continuous variables and simple arithmetic for discrete variables (30) can be transformed into

$$\begin{aligned}W(T_1,T_2) = &\mathop{S}_0^{T_2} V_1(T_1,t_2)\mathrm{d}t_2 - \mathop{S}_0^{T_2} V_1(0,t_2)\mathrm{d}t_2 \\ &+ \mathop{S}_0^{T_1} V_2(t_1,T_2)\mathrm{d}t_1 - \mathop{S}_0^{T_1} V_2(t_1,0)\mathrm{d}t_1.\end{aligned} \tag{31}$$

Since the divergence is non-positive for every t_1 and t_2, from (30) we get $W(T_1,T_2) \le 0$. Also, $V_2(t_1,t_2)$ is a non-negative function of t_1 and t_2. Therefore (31) implies (28). The bound on the integral of $V_2(t_1,t_2)$ in (29) follows equivalently. ■

We now consider the 2D Lyapunov function $\boldsymbol{V}(t_1,t_2)$ introduced above, to show that under some assumptions the system is therefore stable according to Definition 4.

Corollary 1. *Consider the autonomous 2D system in (8). If the following conditions hold*

(i) $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are stable, and
(ii) there exist positive definite, symmetric matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that $\boldsymbol{Q} \le 0$, where $\boldsymbol{Q}$ is given in (18),

then the system is stable as per Definition 4.

Proof: Since $\boldsymbol{A}_{ii}$ is stable, there exist $k_i < \infty$ and $\lambda_i > 0$ (and $\lambda_i < 1$ in the discrete case) such that $\left\|\mathrm{E}(\boldsymbol{A}_{ii})^{t_i}\right\| \le k_i\mathrm{E}(-\lambda_i)^{t_i}$. Therefore, using (9) we have

$$\begin{aligned}|\boldsymbol{x}_1(t_1,t_2)| \le &k_1\mathrm{E}(-\lambda_1)^{t_1}|\boldsymbol{x}_{10}(t_2)| \\ &+ \mathop{S}_0^{t_1} k_1\mathrm{E}(-\lambda_1)^{\tau}\|\boldsymbol{A}_{12}\|\,|\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau,t_2)|\,\mathrm{d}\tau.\end{aligned} \tag{32}$$

We choose $\boldsymbol{P}_2$ as in condition (ii) and then define the Lyapunov function candidate $V_2(t_1,t_2) = \boldsymbol{x}_2^{\mathrm{T}}(t_1,t_2)\,\boldsymbol{P}_2\boldsymbol{x}_2(t_1,t_2)$. Using the definition of $V_2(t_1,t_2)$ and the Cauchy-Schwarz inequality, (32) becomes

$$\begin{aligned}|\boldsymbol{x}_1(t_1,t_2)| \le &k_1|\boldsymbol{x}_{10}(t_2)| + \frac{k_1\|\boldsymbol{A}_{12}\|}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}}\mathop{S}_0^{t_1}\mathrm{E}(-\lambda_1)^{\tau}\sqrt{V_2(t_1 - \mathbb{I}_1 - \tau,t_2)}\mathrm{d}\tau \\ \le &k_1|\boldsymbol{x}_{10}(t_2)| + \frac{k_1\|\boldsymbol{A}_{12}\|}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}} \\ &\cdot\left(\mathop{S}_0^{t_1}\mathrm{E}(-\lambda_1)^{2\tau}\mathrm{d}\tau\right)^{1/2}\left(\mathop{S}_0^{t_1}V_2(\tau,t_2)\,\mathrm{d}\tau\right)^{1/2}.\end{aligned} \tag{33}$$

With (14), Lemma 3 and the fact that the initial conditions are in L_2,

(33) becomes

$$|\boldsymbol{x}_1(t_1,t_2)| \le k_1 |\boldsymbol{x}_{10}(t_2)| + \frac{k_1\|\boldsymbol{A}_{12}\|}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}} \sqrt{\frac{1-\mathrm{E}(-\lambda_1)^{2t_1}}{2\lambda_1 - \lambda_1^2 \mathbb{I}_1}} \cdot \left(\mathcal{S}_0^{t_2} V_1(0,\tau)\mathrm{d}\tau + \mathcal{S}_0^{t_1} V_2(\tau,0)\mathrm{d}\tau \right)^{1/2} \le k_1 |\boldsymbol{x}_{10}(t_2)| + \frac{k_1\|\boldsymbol{A}_{12}\| \sqrt{\|\boldsymbol{P}_1\|c_1 + \|\boldsymbol{P}_2\|c_2}}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}\sqrt{2\lambda_1 - \lambda_1^2\mathbb{I}_1}}. \quad (34)$$

Note that since for t_i discrete we have $\mathbb{I}_i = 1$ and $\lambda_i < 1$ we find that $2\lambda_i - \lambda_i^2\mathbb{I}_i > \lambda_i$. Thus, $1/(2\lambda_i - \lambda_i^2\mathbb{I}_i) < 1/\lambda_i$. Since the initial conditions are also in L_∞, we find that

$$|\boldsymbol{x}_1(t_1,t_2)| \le M_1 =: k_1\zeta_1 + \frac{k_1\|\boldsymbol{A}_{12}\| \sqrt{\|\boldsymbol{P}_1\|c_1 + \|\boldsymbol{P}_2\|c_2}}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}\sqrt{\lambda_1}} \quad (35)$$

for all $t_1,t_2 > 0$. Note that the bound M_1 is scaled by the L_2 and L_∞ norms of the initial conditions, i. e. ζ_1,c_1,c_2. A similar bound for $\boldsymbol{x}_2(t_1,t_2)$ can be found in the same way. The system is therefore stable. ■

Under the same assumptions as in Corollary 1 we can further show that not only is $\boldsymbol{x}_i(t_1,t_2)$ bounded (that is in L_∞) but also the generalised integrals $\mathcal{S}_0^\infty |\boldsymbol{x}_1(t_1,t_2)|^2\mathrm{d}t_1$ and $\mathcal{S}_0^\infty |\boldsymbol{x}_2(t_1,t_2)|^2\mathrm{d}t_2$ are bounded. This will facilitate the proof of asymptotic stability later in Section IV.

Corollary 2. *Consider the autonomous 2D System in (8). If the following conditions hold*

(i) the initial conditions are L_2 and L_∞ bounded according to Definition 2,

(ii) $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are stable, and

(iii) there exist positive definite, symmetric matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that $\boldsymbol{Q} \le 0$, where $\boldsymbol{Q}$ is given in (18),

then there exist $\overline{M}_1, \overline{M}_2 < \infty$ independently of t_2 and t_1, respectively, such that

$$\mathcal{S}_0^\infty |\boldsymbol{x}_1(t_1,t_2)|^2\,\mathrm{d}t_1 \le \overline{M}_1 \quad \text{and} \quad \mathcal{S}_0^\infty |\boldsymbol{x}_2(t_1,t_2)|^2\,\mathrm{d}t_2 \le \overline{M}_2. \quad (36)$$

Proof: From (32), note that

$$\mathcal{S}_0^\infty |\boldsymbol{x}_1(t_1,t_2)|^2\,\mathrm{d}t_1 \le 2k_1^2 \mathcal{S}_0^\infty \mathrm{E}(-\lambda_1)^{2t_1} |\boldsymbol{x}_{10}(t_2)|^2\,\mathrm{d}t_1 + 2k_1^2\|\boldsymbol{A}_{12}\|^2 \mathcal{S}_0^\infty \left(\mathcal{S}_0^{t_1} \mathrm{E}(-\lambda_1)^\tau |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau,t_2)|\mathrm{d}\tau \right)^2 \mathrm{d}t_1. \quad (37)$$

The first term of the right hand side of (37) can be bounded as follows

$$2k_1^2 \mathcal{S}_0^\infty \mathrm{E}(-\lambda_1)^{2t_1} |\boldsymbol{x}_{10}(t_2)|^2\,\mathrm{d}t_1 \le \frac{2k_1^2\zeta_1^2}{\lambda_1}. \quad (38)$$

With the Cauchy-Schwarz inequality the second term of the right hand side of (37) allows a bound to be calculated as

$$2k_1^2\|\boldsymbol{A}_{12}\|^2 \mathcal{S}_0^\infty \left(\mathcal{S}_0^{t_1} \mathrm{E}(-\lambda_1)^\tau |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau,t_2)|\mathrm{d}\tau \right)^2 \mathrm{d}t_1 \le 2k_1^2\|\boldsymbol{A}_{12}\|^2 \mathcal{S}_0^\infty \left(\mathcal{S}_0^{t_1} \mathrm{E}(-\lambda_1)^\tau \mathrm{d}\tau \right) \left(\mathcal{S}_0^{t_1} \mathrm{E}(-\lambda_1)^\tau |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau,t_2)|^2\mathrm{d}\tau \right) \mathrm{d}t_1 \le \frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1} \mathcal{S}_0^\infty \mathcal{S}_0^{t_1} \mathrm{E}(-\lambda_1)^\tau |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau,t_2)|^2\,\mathrm{d}\tau\mathrm{d}t_1. \quad (39)$$

Interchanging the order of integration in (39) yields

$$\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1} \mathcal{S}_0^\infty \mathcal{S}_0^{t_1} \mathrm{E}(-\lambda_1)^{t_1 - \mathbb{I}_1 - \tau} |\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}\tau\mathrm{d}t_1 \le \frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1} \mathcal{S}_0^\infty \mathcal{S}_{\tau+\mathbb{I}_1}^\infty \mathrm{E}(-\lambda_1)^{t_1 - \mathbb{I}_1 - \tau} |\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}t_1\mathrm{d}\tau \le \frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1^2} \mathcal{S}_0^\infty |\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}\tau. \quad (40)$$

Taking the limit as $T_1 \to \infty$ of (29) in Lemma 3 we see that the generalised integral in (40) is bounded independently of t_2. Thus $\overline{M}_1$ exists. The existence of $\overline{M}_2$ follows similarly. ■

To facilitate the proof of asymptotic stability of 2D systems in Section IV we also need results on the state derivatives and will show that under suitable assumptions the first generalised derivatives, i.e. $\delta_i\boldsymbol{x}_k(t_1,t_2)$, $i,k \in \{1,2\}$, are in both L_2 $[0,\infty)\times[0,\infty)$ and L_∞ $[0,\infty)\times[0,\infty)$ and the second generalised derivatives, i.e. $\delta_i\delta_k\boldsymbol{x}_k(t_1,t_2)$ for $i,k \in \{1,2\}$, are in L_∞ $[0,\infty)\times[0,\infty)$.

Lemma 4. *Consider the autonomous 2D System in (8). If the following conditions hold*

(i) the initial conditions are L_2' and L_∞'' smooth bounded according to Definition 3,

(ii) $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are stable, and

(iii) there exist positive definite, symmetric matrices $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{R}$ such that $\boldsymbol{Q} = -\boldsymbol{A}^\mathrm{T}\boldsymbol{R}\boldsymbol{A} \le 0$, where $\boldsymbol{Q}$ is given in (18),

then

(a) the first generalised derivatives of $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ are in L_∞ $[0,\infty)\times[0,\infty)$ and L_2 $[0,\infty)\times[0,\infty)$, i.e. there exist $M_{ik},\overline{M}_{ik} < \infty$ such that for $i,k \in \{1,2\}$,

$$\sup_{(t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2} |\delta_k\boldsymbol{x}_i(t_1,t_2)| \le M_{ik} \quad (41)$$

$$\mathcal{S}_0^\infty \mathcal{S}_0^\infty |\delta_k\boldsymbol{x}_i(t_1,t_2)|^2\mathrm{d}t_1\mathrm{d}t_2 \le \overline{M}_{ik}, \text{ and} \quad (42)$$

(b) the second generalised derivatives of $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ are in L_∞ $[0,\infty)\times[0,\infty)$, i.e. there exist $M_{ikl} < \infty$ such that for $i,k,l \in \{1,2\}$

$$\sup_{(t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2} |\delta_k\delta_l\boldsymbol{x}_i(t_1,t_2)| \le M_{ikl}. \quad (43)$$

Proof: (a): We will first prove that $\delta_1\boldsymbol{x}_1(t_1,t_2)$ (and $\delta_2\boldsymbol{x}_2(t_1,t_2)$) is in L_∞ $[0,\infty)\times[0,\infty)$. Using the state space description for $\delta_1\boldsymbol{x}_1(t_1,t_2)$ in (8) we have

$$|\delta_1\boldsymbol{x}_1(t_1,t_2)| \le \|\boldsymbol{A}_{11}\| \cdot |\boldsymbol{x}_1(t_1,t_2)| + \|\boldsymbol{A}_{12}\| \cdot |\boldsymbol{x}_2(t_1,t_2)|. \quad (44)$$

Since $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ are stable (Corollary 1), there exist $M_1,M_2 < \infty$ such that $|\boldsymbol{x}_i(t_1,t_2)| \le M_i$ for all t_1,t_2 and $i \in \{1,2\}$. Thus, $M_{11} = \|\boldsymbol{A}_{11}\|M_1 + \|\boldsymbol{A}_{12}\|M_2$ and $M_{22} = \|\boldsymbol{A}_{21}\|M_1 + \|\boldsymbol{A}_{22}\|M_2$.

To show that $\delta_2\boldsymbol{x}_1(t_1,t_2)$ and $\delta_1\boldsymbol{x}_2(t_1,t_2)$ are in L_∞ $[0,\infty)\times[0,\infty)$ as well, we operate on (9) by δ_2 and obtain the bound

$$|\delta_2\boldsymbol{x}_1(t_1,t_2)| \le k_1\mathrm{E}(-\lambda_1)^{t_1}|\delta_2\boldsymbol{x}_{10}(t_2)| + \| + k_1\|\boldsymbol{A}_{12} \left| \mathcal{S}_0^{t_1} \mathrm{E}(-\lambda_1)^\tau \delta_2\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau,t_2)\mathrm{d}\tau \right| \le k_1\zeta_1' + \frac{k_1\|\boldsymbol{A}_{12}\|M_{22}}{\lambda_1} =: M_{12}. \quad (45)$$

The boundedness of $\delta_1\boldsymbol{x}_2(t_1,t_2)$ can be proven in the same way.

To show that the first generalised derivatives are also in L_2 $[0,\infty)\times[0,\infty)$ we will use the Lyapunov function candidate $\boldsymbol{V}(t_1,t_2)$ from (16). Given the fact that $\boldsymbol{x}^\mathrm{T}(t_1,t_2)\boldsymbol{Q}\boldsymbol{x}(t_1.t_2)$ is the divergence of $\boldsymbol{V}(t_1,t_2)$ we can show with the fundamental theorem of calculus that

$$\mathcal{S}_0^{T_2}\mathcal{S}_0^{T_1} \begin{bmatrix} \delta_1\boldsymbol{x}_1^\mathrm{T}(t_1,t_2) & \delta_2\boldsymbol{x}_2^\mathrm{T}(t_1,t_2) \end{bmatrix} \boldsymbol{R} \begin{bmatrix} \delta_1\boldsymbol{x}_1(t_1,t_2) \\ \delta_2\boldsymbol{x}_2(t_1,t_2) \end{bmatrix} \mathrm{d}t_1\mathrm{d}t_2 \le \mathcal{S}_0^{T_2} V_1(0,t_2)\mathrm{d}t_2 + \mathcal{S}_0^{T_1} V_2(t_1,0)\mathrm{d}t_1 \quad (46)$$

Taking the limit of both sides of (46) as $T_1,T_2 \to \infty$ we see that

$$\mathcal{S}_0^\infty\mathcal{S}_0^\infty |\delta_1\boldsymbol{x}_1(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2 \le \frac{\|\boldsymbol{P}_1\|c_1 + \|\boldsymbol{P}_2\|c_2}{\sigma_{\min}(\boldsymbol{R})} =: \overline{M}_{11}, \quad (47)$$

$$\mathcal{S}_0^\infty\mathcal{S}_0^\infty |\delta_2\boldsymbol{x}_2(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2 \le \frac{\|\boldsymbol{P}_1\|c_1 + \|\boldsymbol{P}_2\|c_2}{\sigma_{\min}(\boldsymbol{R})} =: \overline{M}_{22}. \quad (48)$$

To show the existence of $\overline{M}_{12}$ we will transform the solution given in (9) into

$$\begin{aligned}&\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty|\delta_2\boldsymbol{x}_1(t_1,t_2)|^2\mathrm{d}t_1\mathrm{d}t_2\\ &\le 2k_1^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty \mathrm{E}(-\lambda_1)^{2t_1}|\delta_2\boldsymbol{x}_{10}(t_2)|^2\mathrm{d}t_1\mathrm{d}t_2+2k_1^2\|\boldsymbol{A}_{12}\|^2\\ &\cdot\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\left|\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}\delta_2\boldsymbol{x}_2(\tau,t_2)\mathrm{d}\tau\right|^2\mathrm{d}t_1\mathrm{d}t_2. \end{aligned}\tag{49}$$

Since the initial conditions are L_2' smooth the first term on the right side of (49) can be bounded by

$$2k_1^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty \mathrm{E}(-\lambda_1)^{2t_1}|\delta_2\boldsymbol{x}_{10}(t_2)|^2\mathrm{d}t_1\mathrm{d}t_2\le\frac{2k_1^2c_1'}{\lambda_1}.\tag{50}$$

The second term can be transformed using the Cauchy Schwarz inequality

$$\begin{aligned}&2k_1^2\|\boldsymbol{A}_{12}\|^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\left|\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}\delta_2\boldsymbol{x}_2(\tau,t_2)\mathrm{d}\tau\right|^2\mathrm{d}t_1\mathrm{d}t_2\\ &\le 2k_1^2\|\boldsymbol{A}_{12}\|^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\left(\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{\tau}\mathrm{d}\tau\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}|\delta_2\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}\tau\right)\mathrm{d}t_1\mathrm{d}t_2.\end{aligned}\tag{51}$$

We will now solve the first inner generalised integral and change the order of (generalised) integration of the remaining part. Thus (51) becomes

$$\begin{aligned}&2k_1^2\|\boldsymbol{A}_{12}\|^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\left|\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}\delta_2\boldsymbol{x}_2(\tau,t_2)\mathrm{d}\tau\right|^2\mathrm{d}t_1\mathrm{d}t_2\\ &\le\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1}\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_{\tau_1+\mathbb{I}_1}^\infty\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}|\delta_2\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}t_1\mathrm{d}\tau\mathrm{d}t_2\\ &\le\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1^2}\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty|\delta_2\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}\tau\mathrm{d}t_2\\ &\le\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1^2}\overline{M}_{22}=:\overline{M}_{12}.\end{aligned}\tag{52}$$

(b): To complete the proof we will show that the second generalised derivatives are in $L_\infty\,[0,\infty)\times[0,\infty)$. First the norm of the generalised derivatives $\delta_1^2\boldsymbol{x}_1(t_1,t_2)$ and $\delta_1\delta_2\boldsymbol{x}_1(t_1,t_2)$ will be considered. Taking the generalised derivative of the first part of the state space description (8) with respect to t_1 or t_2, respectively yields

$$\delta_1^2\boldsymbol{x}_1(t_1,t_2)=\boldsymbol{A}_{11}\delta_1\boldsymbol{x}_1(t_1,t_2)+\boldsymbol{A}_{12}\delta_1\boldsymbol{x}_2(t_1,t_2)\tag{53}$$

$$\delta_1\delta_2\boldsymbol{x}_1(t_1,t_2)=\boldsymbol{A}_{11}\delta_2\boldsymbol{x}_1(t_1,t_2)+\boldsymbol{A}_{12}\delta_2\boldsymbol{x}_2(t_1,t_2)\tag{54}$$

Thus $M_{111}=\|\boldsymbol{A}_{11}\|M_{11}+\|\boldsymbol{A}_{12}\|M_{12}$ and $M_{112}=M_{121}=\|\boldsymbol{A}_{11}\|M_{12}+\|\boldsymbol{A}_{12}\|M_{22}$. To show that $|\delta_2^2\boldsymbol{x}_1(t_1,t_2)|$ is bounded, follow a similar argument as in (45), so that M_{122} becomes

$$M_{122}=k_1\zeta_1''+\frac{k_1\|\boldsymbol{A}_{12}\|M_{222}}{\lambda_1}.\tag{55}$$

The existence of $M_{211}, M_{212}, M_{221}$ and M_{222} can be prooven in the same manner. ■

We will now prove a 2D version of Barbalat's Lemma, [32, Lemma 3.1], which will enable the proof of asymptotic stability of 2D systems.

Lemma 5. *Consider the 2D function $f:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$. If $f(t_1,t_2)$ is both in $L_p\,[0,\infty)\times[0,\infty)$ and $L_\infty\,[0,\infty)\times[0,\infty)$ and both its generalised derivatives $\delta_1 f(t_1,t_2)$ and $\delta_2 f(t_1,t_2)$ are in $L_\infty\,[0,\infty)\times[0,\infty)$, then $\lim_{t_1,t_2\to\infty}f(t_1,t_2)=0$ and $f(t_1,t_2)$ is uniformly convergent in both directions, i.e. for all $\epsilon>0$ there exists a $T(\epsilon)<\infty$ such that*

$$\forall(t_1,t_2)\in\{\mathbb{T}_1\times[T(\epsilon),\infty)\}\cup\{[T(\epsilon),\infty)\times\mathbb{T}_2\}:|f(t_1,t_2)|<\epsilon.$$

Proof: Define the supremum of $f(t_1,t_2)$ and the supremum over the maximum of both generalised derivatives in the complete quadrant as

$$\overline{f}:=\sup_{t_1,t_2\in\mathbb{T}_1\times\mathbb{T}_2}|f(t_1,t_2)|\quad\text{and}\tag{56}$$

$$\overline{f'}:=\sup_{t_1,t_2\in\mathbb{T}_1\times\mathbb{T}_2}\{\max\{|\delta_1 f(t_1,t_2)|,|\delta_2 f(t_1,t_2)|\}\}\tag{57}$$

and the region R_l as

$$R_l:=\{[0,l+1)\times[l,l+1)\}\cup\{[l,l+1)\times[0,l)\}.\tag{58}$$

Note then that

$$\|f(\cdot,\cdot)\|^p_{L_p\,[0,\infty)\times[0,\infty)}=\sum_{l=0}^{\infty}\mathop{\mathcal{S}\mathcal{S}}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2<\infty\tag{59}$$

where $\mathcal{S}\mathcal{S}_{R_l}\cdot\mathrm{d}t_1\mathrm{d}t_2$ refers to the 2D integration over the region R_l. Therefore,

$$\lim_{l\to\infty}\mathop{\mathcal{S}\mathcal{S}}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2=0.\tag{60}$$

Let the supremum of f within R_l be defined as

$$\overline{f_l}:=\sup_{(t_1,t_2)\in R_l}|f(t_1,t_2)|.\tag{61}$$

Then if t_1 is continuous

$$\sup_{(t_1,t_2)\in R_l}\frac{\mathrm{d}}{\mathrm{d}t_1}|f(t_1,t_2)|^p\le\sup_{(t_1,t_2)\in R_l}\left(p|f(t_1,t_2)|^{p-1}\left|\frac{\mathrm{d}}{\mathrm{d}t_1}f(t_1,t_2)\right|\right)\le p\overline{f_l}^{p-1}\overline{f'}.\tag{62}$$

We will now bound the double generalised integral $\mathcal{S}\mathcal{S}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2$ from below using the geometric form of $f(t_1,t_2)$ depending on the nature of t_1 and t_2.

If both independent variables t_1 and t_2 are continuous, $\mathcal{S}\mathcal{S}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2$ is the double integral over an L-shaped surface. It can be bounded from below by the smallest possible pyramid with height $\overline{f_l}^p$, where the base is bounded by $\frac{\overline{f_l}}{p\overline{f'}}$ or the dimensions of the region R_l.

In case one variable is continuous and one is discrete (mixed case) $\mathcal{S}\mathcal{S}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2$ is a summation of l line integrals. It can be bounded from below by the smallest possible triangle with height $\overline{f_l}^p$, where the base is bounded by $\frac{\overline{f_l}}{p\overline{f'}}$ or the smallest possible length of any line fragment in R_l.

If both variables are discrete $\mathcal{S}\mathcal{S}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2$ is a summation with $2l+1$ summands. Thus it can be bounded from below by a single summand. (Here we will take the maximal summand $\overline{f_l}^p$.)

$$\begin{aligned}&\mathop{\mathcal{S}\mathcal{S}}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2\\ &\ge\begin{cases}\frac{1}{6}\overline{f_l}^p\min\left\{\frac{\overline{f_l}}{p\overline{f'}},l+1\right\}\min\left\{\frac{\overline{f_l}}{p\overline{f'}},1\right\}: & t_1,t_2\text{ continuous}\\ \frac{1}{2}\overline{f_l}^p\min\left\{\frac{\overline{f_l}}{p\overline{f'}},1\right\}: & \text{mixed case}\\ \overline{f_l}^p: & t_1,t_2\text{ discrete}\end{cases}\end{aligned}\tag{63}$$

If both t_1 and t_2 are discrete the result follows immediately from (60). In the continuous case we can transform (63) into

$$\begin{aligned}\mathop{\mathcal{S}\mathcal{S}}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2\ge&\frac{1}{6}\overline{f_l}^p\min\left\{\frac{\overline{f_l}}{p\overline{f'}},(l+1)\frac{\overline{f_l}}{\overline{f}}\right\}\min\left\{\frac{\overline{f_l}}{p\overline{f'}},\frac{\overline{f_l}}{\overline{f}}\right\}\\ =&\frac{1}{6}\overline{f_l}^{p+2}\min\left\{\frac{1}{p\overline{f'}},\frac{l+1}{\overline{f}}\right\}\min\left\{\frac{1}{p\overline{f'}},\frac{1}{\overline{f}}\right\}\end{aligned}\tag{64}$$

Thus

$$\begin{aligned}\overline{f_l}^{p+2}\le&6\max\left\{p\overline{f'},\frac{\overline{f}}{l+1}\right\}\max\left\{p\overline{f'},\overline{f}\right\}\mathop{\mathcal{S}\mathcal{S}}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2\\ \le&6\left(\max\left\{p\overline{f'},\overline{f}\right\}\right)^2\mathop{\mathcal{S}\mathcal{S}}_{R_l}|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2\end{aligned}\tag{65}$$

As $\overline{f'}$ and $\overline{f}$ are bounded $\overline{f_l}$ tends to zero as l grows without bound. Hence from the definition of $\overline{f_l}$ (61), $f(t_1,t_2)$ for $(t_1,t_2) \in R_l$ tends to zero as l grows without bound.

In case one variable is continuous and one is discrete a similar argument can be made. ■

IV. Asymptotic Stability

In this section we will present our theorem on asymptotic stability of 2D systems described by the Roesser model using intermediate results presented in the previous section.

Theorem 1 (Asymptotic Stability of 2D Roesser Models)**.** *The 2D system (8) is asymptotically stable with smooth bounded initial conditions according to Definition 5 if the following conditions hold*

(i) $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are stable, and

(ii) there exist positive definite, symmetric matrices $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{R}$ such that $\boldsymbol{Q} = -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}$, where $\boldsymbol{Q}$ is given in (18).

Proof: Consider the 2D Lyapunov function $\boldsymbol{V}(t_1,t_2)$ given in (16) and the integral of $V_1(t_1,t_2) + V_2(t_1,t_2)$ along the line $\Omega(l) := (t_1,t_2) \in \{[0,l] \times \{l\}\} \cup \{\{l\} \times [0,l]\}$ for $l \in \mathbb{R}^+$ or $l \in \mathbb{N}$, respectively, and $l > 0$ as:

$$U(l) := \underset{\Omega(l)}{\mathcal{S}} (V_1(t_1,t_2) + V_2(t_1,t_2))\,\mathrm{d}s$$
$$= \overset{l}{\underset{0}{\mathcal{S}}} (V_1(t_1,l) + V_2(t_1,l))\,\mathrm{d}t_1 + \overset{l}{\underset{0}{\mathcal{S}}} (V_1(l,t_2) + V_2(l,t_2))\,\mathrm{d}t_2 \qquad (66)$$

Using the results in Lemma 3 and Corollary 2 we see that there exists a C such that for all l: $U(l) \le C$. Since the first generalised derivatives of $\boldsymbol{x}(t_1,t_2)$ with respect to t_1 and t_2 are L_∞ bounded (Lemma 4) we can find $d_{11}(l)$, $d_{12}(l)$, $d_{21}(l)$ and $d_{22}(l)$ such that

$$d_{11}(l) := \sup_{0\le t_1 \le l} |\delta_1 \boldsymbol{x}_1(t_1,l)|_2\,, \qquad (67)$$

$$d_{12}(l) := \sup_{0\le t_2 \le l} |\delta_2 \boldsymbol{x}_1(l,t_2)|_2\,, \qquad (68)$$

$$d_{21}(l) := \sup_{0\le t_1 \le l} |\delta_1 \boldsymbol{x}_2(t_1,l)|_2\,, \qquad (69)$$

$$d_{22}(l) := \sup_{0\le t_2 \le l} |\delta_2 \boldsymbol{x}_2(l,t_2)|_2\,. \qquad (70)$$

Note that $d_{11}(l) \le \sup_{t_1\ge 0} |\delta_1 \boldsymbol{x}_1(t_1,l)|_2$. Making use of the version of Barbalat's Lemma in Lemma 5, we can conclude that the first generalised derivatives tend to zero as $t_1,t_2 \to \infty$ and are uniformly convergent in both directions. That allows us to interchange the order of supremum and limit and thus we conclude that

$$\lim_{l\to\infty} d_{11}(l) \le \lim_{l\to\infty} \sup_{t_1\ge 0} |\delta_1 \boldsymbol{x}_1(t_1,l)|_2$$
$$= \sup_{t_1\ge 0} \lim_{l\to\infty} |\delta_1 \boldsymbol{x}_1(t_1,l)|_2$$
$$= 0. \qquad (71)$$

It can be shown in a similar way that the limits of $d_{12}(l)$, $d_{21}(l)$, and $d_{22}(l)$ for $l \to \infty$ are 0.

Thus for t_1,t_2 continuous we can bound the derivatives of $V_1(t_1,t_2)$ and $V_2(t_1,t_2)$ by

$$\forall t_1 \le l : \frac{\mathrm{d}}{\mathrm{d}t_1} V_1(t_1,l) \le 2d_{11}(l)\|\boldsymbol{P}_1\|M_1, \qquad (72)$$

$$\forall t_2 \le l : \frac{\mathrm{d}}{\mathrm{d}t_2} V_1(l,t_2) \le 2d_{12}(l)\|\boldsymbol{P}_1\|M_1, \qquad (73)$$

$$\forall t_1 \le l : \frac{\mathrm{d}}{\mathrm{d}t_1} V_2(t_1,l) \le 2d_{21}(l)\|\boldsymbol{P}_2\|M_2, \qquad (74)$$

$$\forall t_2 \le l : \frac{\mathrm{d}}{\mathrm{d}t_2} V_2(l,t_2) \le 2d_{22}(l)\|\boldsymbol{P}_2\|M_2, \qquad (75)$$

where M_1 and M_2 are bounds on $|\boldsymbol{x}_1(t_1,t_2)|$ and $|\boldsymbol{x}_2(t_1,t_2)|$ (as introduced in the proof of Lemma 4). Note that in fact the same bounds apply for t_1 or t_2 discrete because for t_1 discrete we have for $(t_1,t_2) \in \Omega(l)$

$$\delta_1\left(\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}_1\right) \le \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}_1 - (\boldsymbol{x}_1 - d_{11}(l)\boldsymbol{1})^{\mathrm{T}}\boldsymbol{P}_1(\boldsymbol{x}_1 - d_{11}(l)\boldsymbol{1})$$
$$\le 2d_{11}(l)\|\boldsymbol{P}_1\|M_1 \qquad (76)$$

where $\boldsymbol{1}$ is a vector of 1s of appropriate length.

To find a lower bound on $U(l)$ we will use a similar trick as in the proof of Lemma 5 above.

If t_1 is continuous and the maximum of $V_i(t_1,t_2)$ ($\overline{V}_i(l) := \max_{(t_1,t_2)\in\Omega(l)} V_i(t_1,t_2)$ for $i \in \{1,2\}$) along $\Omega(l)$ occurs along the part of $\Omega(l)$ where $(t_1,t_2) \in [0,l] \times \{l\}$ we can bound the integral of $V_i(t_1,t_2)$ over $\Omega(l)$ from below by a triangle with the base equal to $\min\left\{\overline{V}_i(l)\big/(2d_{i1}(l)\|\boldsymbol{P}_i\|M_i), l\right\}$ and $\overline{V}_i(l)$ as the height of the triangle. In case t_1 is discrete and $\overline{V}_i(l)$ occurs at $(t_1,t_2) \in \{l\}\times[0,l]$ the summation of $V_i(t_1,t_2)$ along t_1 for $l > \overline{V}_i(l)/2d_{i1}(l)\|\boldsymbol{P}_i\|M_i$ can be bounded by

$$\sum_{t_1=0}^{l} V_i(t_1,t_2) \ge \overline{V}_i(l) + \left(\overline{V}_i(l) - 2d_{i1}(l)\|\boldsymbol{P}_i\|M_i\right)$$
$$+ \left(\overline{V}_i(l) - 4d_{i1}(l)\|\boldsymbol{P}_i\|M_i\right)\ldots$$
$$= (\nu+1)\overline{V}_i(l) - 2d_{i1}(l)\|\boldsymbol{P}_i\|M_i \sum_{n=1}^{\nu} n \qquad (77)$$

where $\nu = \left\lfloor \overline{V}_i(l)/2d_{i1}(l)\|\boldsymbol{P}_i\|M_i \right\rfloor$. Resolving the summation on the right hand side of (77) yields

$$\sum_{t_1=0}^{l} V_i(t_1,t_2) \ge (\nu+1)\left(\overline{V}_i(l) - 2d_{i1}(l)\|\boldsymbol{P}_i\|M_i \frac{\nu}{2}\right)$$
$$\ge (\nu+1)\left(\overline{V}_i(l) - 2d_{i1}(l)\|\boldsymbol{P}_i\|M_i \frac{\overline{V}_i(l)/2d_{i1}(l)\|\boldsymbol{P}_i\|M_i}{2}\right)$$
$$= (\nu+1)\frac{\overline{V}_i(l)}{2}$$
$$\ge \frac{\overline{V}_i^2(l)}{4d_{i1}(l)\|\boldsymbol{P}_i\|M_i}. \qquad (78)$$

For $l \le \overline{V}_i(l)/2d_{i1}(l)\|\boldsymbol{P}_i\|M_i$ the summation can be bounded by

$$\sum_{t_1=0}^{l} V_i(t_1,t_2) \ge \frac{\overline{V}_i(l)l}{2}. \qquad (79)$$

Thus, $U(l)$ can be bounded

$$U(l) \ge \min\left\{\frac{\overline{V}_1^2(l)}{4d_{11}(l)\|\boldsymbol{P}_1\|M_1}, \frac{\overline{V}_1^2(l)}{4d_{12}(l)\|\boldsymbol{P}_1\|M_1}, \frac{\overline{V}_1(l)l}{2}\right\}$$
$$+ \min\left\{\frac{\overline{V}_2^2(l)}{4d_{21}(l)\|\boldsymbol{P}_2\|M_2}, \frac{\overline{V}_2^2(l)}{4d_{22}(l)\|\boldsymbol{P}_2\|M_2}, \frac{\overline{V}_2(l)l}{2}\right\}. \qquad (80)$$

Since $\overline{V}_i(l) \le M_i^2\|\boldsymbol{P}_i\|$ this implies

$$\overline{V}_i^2(l) \le C \max\left\{4d_{i1}(l)\|\boldsymbol{P}_i\|M_i, 4d_{i2}(l)\|\boldsymbol{P}_i\|M_i, \frac{2M_i^2\|\boldsymbol{P}_i\|}{l}\right\} \qquad (81)$$

Note that as l tends to infinity each component of the maximum in (81) goes to zero and, hence, $\lim_{t_1,t_2\to\infty} |\boldsymbol{x}_i(t_1,t_2)| = 0$. Note that the limits $\lim_{t_1\to\infty} |\boldsymbol{x}_i(t_1,t_2)| = 0$ and $\lim_{t_2\to\infty} |\boldsymbol{x}_i(t_1,t_2)| = 0$ exist as well. ■

V. Examples

To illustrate our result on asymptotic stability of 2D systems we will discuss a simple platooning problem (Section I-C). As discussed in Section I-C every such 2D system describing a vehicle platoon includes a singularity at $s = 0$ and $z = 1$. Thus, there does not exist a matrix $\boldsymbol{P}$ such that $\boldsymbol{Q} < 0$. (Note that relaxing the restriction to quadratic Lyapunov functions does not alter this conclusion.)

Example 1. We will use a simplified, linearised second order model for each vehicle with $P(s) = 1/s(s+2C_\mathrm{d}v_0)$ where the drag coefficient is $C_\mathrm{d} = 7\cdot 10^{-4}m^{-1}$. We choose a simple PID controller with $k_\mathrm{p} = 1.66$, $k_\mathrm{i} = 0.17$, $k_\mathrm{d} = 4.1$ and $T = 1/30$ to minimize the local spacing error. (A more detailed discussion of the system can be found in [33, p. 7].) It can be shown that using a fixed distance policy will lead to string instability or the 'slinky effect', where disturbances are attenuated while traveling through the string, [33].

One possibility, [34], to avert string instability is to introduce a time headway h and maintain a velocity depending distance between each vehicle and its predecessor rather than a fixed distance. In order to maintain the same closed loop poles of the kth vehicle an additional pole at $-\frac{1}{h}$ is added to each local controller. Thus the system can be described as a 2D Roesser model as $\delta\boldsymbol{x}(t,k) = \boldsymbol{A}\boldsymbol{x}(t,k)$ with

$$\boldsymbol{A} = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2C_\mathrm{d}v_0 & 1 & 0 & 0 & 0 \\ -\frac{1}{h}\left(k_\mathrm{p}+\frac{k_\mathrm{d}}{T}\right) & -\left(k_\mathrm{p}+\frac{k_\mathrm{d}}{T}\right) & -\frac{1}{h} & \frac{1}{h} & -\frac{k_\mathrm{d}}{hT^2} & \frac{1}{h}\left(k_\mathrm{p}+\frac{k_\mathrm{d}}{T}\right) \\ -k_\mathrm{i} & -hk_\mathrm{i} & 0 & 0 & 0 & 0 \\ -1 & -h & 0 & 0 & -\frac{1}{T} & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \tag{82}$$

where $t_1 = t$ is continuous, $t_2 = k$ is discrete, $\boldsymbol{x}_1(t,k)$ is the state vector of the kth vehicle including its position $\hat{x}(t,k)$, velocity $\hat{v}(t,k)$, and three controller states $\hat{x}_{\mathrm{c}_i}(t,k)$ for $i \in \{1,2,3\}$ and $\boldsymbol{x}_2(t,k)$ is the position of the predecessor to vehicle k at time t, $\hat{x}(t,k-1)$.

It can be shown that choosing a time headway $h > 1.18s$ the system is string stable, [35]. For $h = 2s$ the eigenvalues of the upper left part of $\boldsymbol{A}$ are -25.1, -4.5, -0.5, -0.25 and -0.18. Thus $\boldsymbol{A}_{11}$ is Hurwitz stable. $\boldsymbol{A}_{22} + \mathbf{I} = 0$ is Schur stable.

Matlab finds two symmetric, positive definite matrices:

$$\boldsymbol{P}_1 = \begin{bmatrix} 1.72\cdot 10^3 & 0 & 0 & 5.05\cdot 10^3 & 0 \\ 0 & 2.65\cdot 10^3 & 2.09\cdot 10^3 & -1.69\cdot 10^4 & -1.27\cdot 10^5 \\ 0 & 2.09\cdot 10^3 & 5.92\cdot 10^3 & -2.16\cdot 10^4 & -3.65\cdot 10^5 \\ 5.05\cdot 10^3 & -1.69\cdot 10^4 & -2.16\cdot 10^4 & 1.77\cdot 10^5 & 1.32\cdot 10^6 \\ 0 & -1.27\cdot 20^5 & -3.65\cdot 10^5 & 1.32\cdot 10^6 & 2.25\cdot 10^7 \end{bmatrix}$$

with eigenvalues at $2.26\cdot 10^7$, 10^5, $1.78\cdot 10^3$, 711 and 7.84 and $\boldsymbol{P}_2 = 859$ such that the eigenvalues of $\boldsymbol{Q}$ are $-4.5\cdot 10^6$, $-1.78\cdot 10^4$, $-4.48\cdot 10^3$, -587, -106 and 0. There exists a positive definite matrix

$$\boldsymbol{R}_1 = \begin{bmatrix} 3.01\cdot 10^3 & -7.08\cdot 10^3 & -1.15\cdot 10^4 & 6.98\cdot 10^4 & 7.08\cdot 10^5 \\ -7.08\cdot 10^3 & 2.96\cdot 10^4 & 4.95\cdot 10^4 & -2.95\cdot 10^5 & 3.04\cdot 10^6 \\ -1.15\cdot 10^4 & 4.95\cdot 10^4 & 8.64\cdot 10^4 & -4.93\cdot 10^5 & -5.31\cdot 10^6 \\ 6.98\cdot 10^4 & -2.95\cdot 10^5 & -4.93\cdot 10^5 & 2.96\cdot 10^6 & 3.02\cdot 10^7 \\ 7.08\cdot 10^5 & -3.04\cdot 10^6 & -5.31\cdot 10^6 & 3.02\cdot 10^7 & 3.26\cdot 10^8 \end{bmatrix}$$

and $\boldsymbol{R}_2 = 1$ such that $\boldsymbol{Q} = -\boldsymbol{A}^\mathrm{T}\boldsymbol{R}\boldsymbol{A}$. Thus the system is asymptotically stable in the 2D sense and hence string stable.

Simulation results are displayed in Fig. 1 and Fig. 2. In Fig. 1 we see that the local error signal $\hat{e}(t,k)$ tends to zero for $t \to \infty$. Thus every single subsystem is asymptotically stable. Also the maximum of $\hat{e}(t,k)$ over time and the L_2 norm with respect to time decreases when k grows. See Fig. 2 for details.

After demonstrating an affirming example, where asymptotic stability can be shown using Theorem 1, we will choose two examples, where one condition for asymptotic stability in Theorem 1 is violated each time and the system is not asymptotically stable. In this way we show that there is no trivial relaxation of the conditions for Theorem 1 that produces the same result.

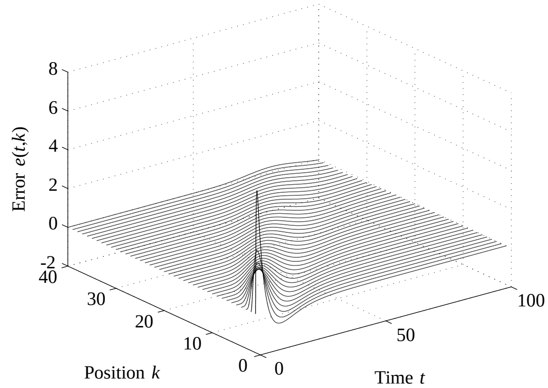

Figure 1: String stable system with $h = 2s$: error $\hat{e}(t,k)$

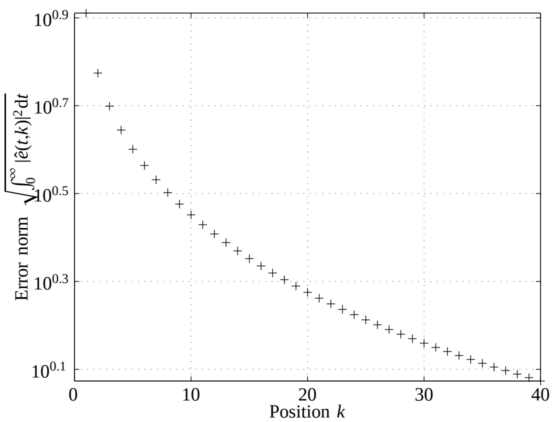

Figure 2: String stable system with $h = 2s$: $\|\hat{e}(\cdot,k)\|_2$

Example 2. Consider the same system structure as presented in Example 1. However, choosing a time headway of $h = 0.5s$ that is clearly less than the infimal time headway required, will lead to a string unstable system.

Even though $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22} + \mathbf{I}$ are Hurwitz and Schur stable, respectively, it is not possible to find a symmetric, positive definite matrix $\boldsymbol{P}$ such that $\boldsymbol{Q} \leq 0$. To show that suppose that there exist $\boldsymbol{P}_1, \boldsymbol{P}_2 > 0$ such that $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$. Without loss of generality we can set $\boldsymbol{P}_2 = 1$. We have

$$\begin{aligned}\boldsymbol{Q} &= \boldsymbol{A}^\mathrm{T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^\mathrm{T}\left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right]\boldsymbol{A} \\ &= \begin{bmatrix} \boldsymbol{A}_{11}\boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{A}_{11} + \boldsymbol{A}_{21}^\mathrm{T}\boldsymbol{A}_{21} & \boldsymbol{P}_1\boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^\mathrm{T}\boldsymbol{P}_1 & -1 \end{bmatrix}\end{aligned} \tag{83}$$

Using the Schur compliment we see that $\boldsymbol{Q} \leq 0$ is equivalent to

$$\boldsymbol{A}_{11}\boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{A}_{11} + \boldsymbol{A}_{21}^\mathrm{T}\boldsymbol{A}_{21} + \boldsymbol{P}_1\boldsymbol{A}_{12}\boldsymbol{A}_{12}^\mathrm{T}\boldsymbol{P}_1 \leq 0 \tag{84}$$

Using the Bounded Real Lemma we see that this is equivalent to

$$\left\|\boldsymbol{A}_{21}\left(s\mathbf{I} - \boldsymbol{A}_{11}\right)^{-1}\boldsymbol{A}_{12}\right\|_\infty \leq 1. \tag{85}$$

Note that $\Gamma(s) = \boldsymbol{A}_{21}\left(s\mathbf{I} - \boldsymbol{A}_{11}\right)^{-1}\boldsymbol{A}_{12}$ is the transfer function from the position of the kth vehicle to the position of the $k+1$th vehicle. However, when choosing a time headway that is less than the infimal headway $h_0 = 1.18s$ we know that $\|\Gamma(j\omega)\|_\infty > 1$. Therefore, generally, any string unstable system of this type ($\|\Gamma(j\omega)\|_\infty > 1$), does not permit a solution with $\boldsymbol{Q} \leq 0$.

In the simulation (displayed in Fig. 3) we observe that the system is not stable in the 2D sense and thus not string stable because a small perturbation at the beginning of the string is amplified while

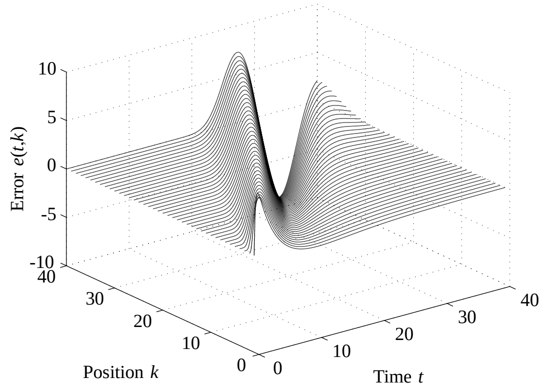

Figure 3: String unstable system with $h = 0.5s$: error $\hat{e}(t,k)$

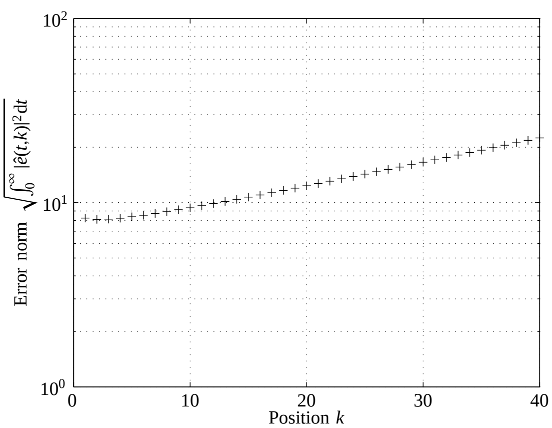

Figure 4: String unstable system with $h = 0.5s$: $||\hat{e}(\cdot,k)||_2$

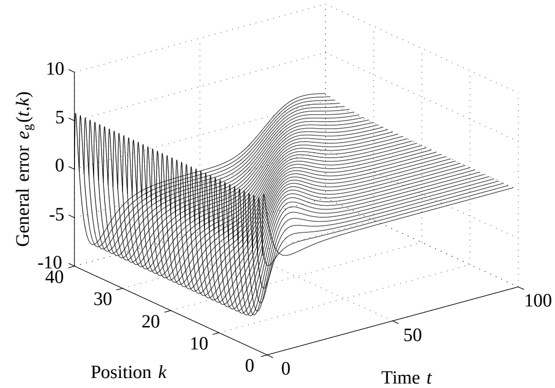

Figure 5: String unstable system: general error $\hat{e}_\mathrm{g}(t,k)$

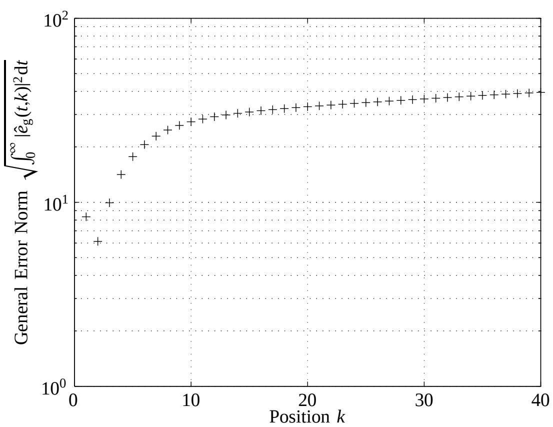

Figure 6: String unstable system: L_2 norm of general error $\hat{e}_\mathrm{g}(t,k)$

traveling through the string. The local error $\hat{e}(t,k)$ goes to zero for every fixed k as $t \to \infty$. However, the maximal error over time for each subsystem grows with k and the double limit $\lim_{t,k\to\infty} \hat{e}(t,k)$ does not exist. Also the L_2 norm of $\hat{e}(t,k)$ with respect to time grows as k grows, Fig. 4.

Also when relaxing the first condition for asymptotic stability in Theorem 1 and allowing $\boldsymbol{A}_{11}$ or $\boldsymbol{A}_{22}$ not to be stable the system might not be asymptotically stable.

Example 3. Consider the system described in (82) with the general error $\hat{e}_\mathrm{g}(t,k)$ (that is the general error of the predecessor plus the local error) as an additional state in $\boldsymbol{x}_2(t,k)$ (and $h = 2s$) such that the system matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \left[\begin{array}{ccccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2C_\mathrm{d}v_0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{h}\left(k_\mathrm{p} + \frac{k_\mathrm{d}}{T}\right) & -\left(k_\mathrm{p} + \frac{k_\mathrm{d}}{T}\right) & -\frac{1}{h} & \frac{1}{h} & -\frac{k_\mathrm{d}}{hT^2} & \frac{1}{h}\left(k_\mathrm{p} + \frac{k_\mathrm{d}}{T}\right) & 0 \\ -k_\mathrm{i} & -hk_\mathrm{i} & 0 & 0 & 0 & k_\mathrm{i} & 0 \\ -1 & -h & 0 & 0 & -\frac{1}{T} & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{86}$$

While $\boldsymbol{A}_{11}$ is still Hurwitz stable, $\boldsymbol{A}_{22} + \mathbf{I}$ has one eigenvalue at 1. Thus it is not Schur stable. The first part of the second condition is not violated as Matlab can find strictly positive matrices $\boldsymbol{P}_1$ with eigenvalues at 130, $7.72 \cdot 10^3$, $2.56 \cdot 10^4$, $7.71 \cdot 10^5$ and $1.58 \cdot 10^8$ and $\boldsymbol{P}_2$ with eigenvalues at $6.66 \cdot 10^3$ and $1.08 \cdot 10^6$ such that $\boldsymbol{Q}$ has eigenvalues at $-1.75 \cdot 10^7$, $-1.22 \cdot 10^5$,$-4.42 \cdot 10^4$, $-3.31 \cdot 10^3$, $-1.25 \cdot 10^3$ and two at 0.

Also $\boldsymbol{A}$ has two eigenvalues at 0 and there exists a positive definite matrix $\boldsymbol{R}$ such that $-\boldsymbol{A}^\mathrm{T}\boldsymbol{R}\boldsymbol{A} = \boldsymbol{Q}$.

So, even though $\boldsymbol{Q}$ is negative semidefinite and there exists a suitable $\boldsymbol{R}$ the system is not asymptotically stable as the simulation in Fig. 5 and Fig. 6 demonstrate.

VI. Conclusions

Here we have discussed stability of 2D systems with two different independent variables. Both continuous and discrete cases have been studied using LMIs and a 2D Lyapunov function.

We showed that a negative semidefinite solution of the Lyapunov function together with additional stability requirements on the systems matrix $\boldsymbol{A}$ is sufficient to guarantee stability and even asymptotic stability under some extra technical conditions of these special 2D systems.

Since only a negative semidefinite solution is required the results presented in this paper are suitable to discuss the stability of 2D systems with singularities on the stability boundary.

In the future we will be interested to relax the requirements on the systems matrix $\boldsymbol{A}$ and the initial conditions and extend the work to non linear systems. Another interesting extension would be the study of suitable controller design techniques.

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