Asymptotic and Exponential Stability of Nonlinear Two-Dimensional Continuous-Discrete Roesser Models

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Abstract

Sufficient conditions guaranteeing Lyapunov stability, asymptotic stability and exponential stability of nonlinear two-dimensional continuous-discrete systems are proposed. Special attention is paid to neutrally stable systems such as some two-dimensional system descriptions of vehicle platoons, which may be stable or asymptotically stable but never exponentially stable. Our conditions for Lyapunov stability and asymptotic stability only require the corresponding two-dimensional Lyapunov function to have a negative semidefinite divergence. They are thus suitable for the analysis of non-exponential versions of 2D stability. Examples are given to illustrate the results.

Keywords: two-dimensional (2D), nonlinear systems, stability, continuous-discrete

1. Introduction

In this paper different notions of stability of two-dimensional (2D) nonlinear systems will be examined. 2D refers to the fact that signals depend on two independent variables. Since both independent variables can be continuous or discrete, most analyses distinguish between discrete-discrete, continuous-continuous and continuous-discrete 2D systems. The majority of the past research on linear systems focuses on discrete-discrete 2D systems due to the range of applications for this case. Compared to the large variety of results on the stability of linear 2D systems, little work seems to be available concerning the stability of nonlinear 2D systems.

1.1. Results on nonlinear 2D systems

Most research appears to focus on particular types of nonlinearities. A range of papers, for example, analyse overflow nonlinearities in general and saturations in particular. See for instance [5, 14, 7, 8, 18, 19, 20, 21, 23, 15]. Sufficient conditions

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for stability based on LMIs for systems with more general nonlinearities in the sector $[0,g_k]$ were shown in [6].

Very few results are available studying general nonlinear 2D systems. Stability of a general nonlinear discrete-discrete 2D system was first analysed in [12]. The authors used a scalar Lyapunov function to guarantee local stability and local and global asymptotic stability. A similar time-varying model was studied in [13].

In [25] the general discrete-discrete 2D Fornasini-Marchesini second model was considered. Using Lyapunov arguments based on the theory of input to state stability (ISS), sufficient conditions for local and global asymptotic stability, in the presence of bounded, decaying initial conditions were derived. Using a scalar Lyapunov function, (global) stability is guaranteed if its 2D difference is nonpositive. If the difference is negative definite, the system is (globally) asymptotically stable. These results were extended in [26] to time (or parameter) varying systems. Sufficient conditions for uniform stability based on the difference being nonpositive are given. In case the difference is less or equal to the negative weighted sum of the state norms, exponential stability can be guaranteed.This is in fact the only result on exponential stability of general 2D discrete-discrete systems known to the authors.

It should be noted that all results of the Lyapunov type for asymptotic stability of nonlinear 2D systems known to the author require the divergence or deviation of the Lyapunov function to be *strictly* negative. Some classes of vehicle platoons can be modelled as 2D continuous-discrete systems. Their special dynamics do not allow simple extension of existing stability results since this class does not admit a Lyapunov function with strictly negative difference or derivative. The study of suitable conditions for stability and asymptotic stability of these systems motivated this research and is further explained in the following subsection.

1.2. A motivating example of nonlinear 2D continuous-discrete systems: string stability of vehicle platoons

There exist classes of asymptotically stable nonlinear 2D continuous-discrete systems that do not admit a Lyapunov function with a strictly negative divergence or difference. One such class arises in vehicle platoons with unidirectional decentralised control. To achieve tight spacing between vehicles travelling in a string (or "platoon"), suppose the vehicles have an automatic controller for longitudinal position. This controller uses local measurements to regulate the distance to the predecessor, or in the case of the lead vehicle, to follow a given trajectory. The overall objective is to achieve "string stability" of the system. This refers to a system where the trajectories remain bounded for all vehicles independently of the position within the string and the string length, i.e. the number of vehicles.

Assume that the system is modelled as a 2D continuous-discrete system driven by the continuous time $t \in \mathbb{R}_{\geq 0}$ and the discrete position within the string $k \in \mathbb{N}$. String stability then corresponds to global Lyapunov stability of the corresponding 2D system as it requires all states to remain bounded for all t and k. Note that, in practise, vehicle platoons consist of a finite number of vehicles N. Every unidirectional string of length N can be seen as a truncation of an infinite string. Since string stability requires the states to be bounded independently of k or N, this is equivalent to stability of the related 2D system. The local state space variables of the kth vehicle (such as position, velocity and controller states) are summarised in the sequence of vector valued functions of time $x_c(t,k) \in \mathbb{R}^{n_c}$. Further, assume the position of the preceding vehicle, that is used as a reference for the kth vehicle, is set to be the scalar $x_d(t,k) \in \mathbb{R}$. The system can thus be modelled similar to the Roesser model as

$$\begin{pmatrix} \dot{x}_c(t,k) \\ \Delta x_d(t,k) \end{pmatrix} = \begin{pmatrix} \frac{d}{dt} x_c(t,k) \\ x_d(t,k+1) - x_d(t,k) \end{pmatrix} = \begin{pmatrix} f_c(x_c(t,k), x_d(t,k)) \\ f_d(x_c(t,k), x_d(t,k)) \end{pmatrix}. \tag{1}$$

The corresponding boundary or initial conditions consist of the initial states of each vehicle in the string $x_c(0,k) = x_{c0}(k)$ and the reference signal $x_d(t,0) = x_{d0}(t)$ that the lead vehicle must follow.

Assume further that when the reference signal is a unit step, the objective is for the positions of each vehicle in the string to approach 1 as $t \to \infty$. Therefore, at best, such systems generically converge to an equilibrium that depends on the boundary conditions, specifically, $x_{d0}(\infty)$ where this limit exists. This kind of behaviour is sometimes referred to as consensus or synchronisation of the individual behaviours. We show below that for important classes of systems, under the assumption that the overall system satisfies the asymptotic consensus condition, it cannot be exponentially stable.

To illustrate this difficulty, consider a class of platoon systems with vehicle dynamics of the kth car described by the homogeneous simple linear model $\dot{x}(t,k) = -ax(t,k) + bx(t,k-1)$. Setting $a = b$ is essential for consensus, that is, to allow the vehicle to follow its predecessor with zero steady state error. The transfer function between the two vehicles is then $G(s) = \frac{a}{s+a}$. Assume further that the initial conditions for all vehicles in the string are zero, i.e. $x(0,k) = 0 \ \forall k$, and the reference trajectory for the first vehicle is $x(t,0) = e^{-at}u(t)$. The trajectory of the kth vehicle is $x(t,k) = \mathcal{L}^{-1}\left\{\frac{a^k}{(s+a)^{k+1}}\right\} = \frac{a^k t^k e^{-at}}{k!}$. To find the local maximum of $x(t,k)$, note that $\frac{d}{dt}x(t,k) = \frac{a^k t^{k-1} e^{-at}}{k!}(k - at)$, which is zero for $t = \frac{k}{a}$. Hence, the value of the local maximum of $x(t,k)$ is $x\left(\frac{k}{a},k\right) = \frac{k^k e^{-k}}{k!}$. Using Stirling's formula, $k! \leq e\sqrt{k}\left(\frac{k}{e}\right)^k$ [3, Fact 1.11.20], it follows that the local maximum of $x(t,k)$ is greater or equal to $\frac{1}{e\sqrt{k}}$. (Fig. 1 illustrates the value of the local maxima vs the position within the string for $a = 1$.) Hence, there exists no exponentially decaying upper bound, that is uniform in k, for the local maxima of $x(t,k)$ and the system cannot be exponentially stable in the 2D sense. Hence, such systems cannot admit a Lyapunov function with strictly negative divergence as this would be sufficient for exponential stability.

Note further that this difficulty is not restricted to systems with first order continuous dynamics. More generally, any linear homogeneous 2D system which satisfies the consensus property, must have a 2D characteristic polynomial with a pole locus that touches the stability boundary at $s = 0$, $z = 1$ (denoted "singularity on the stability boundary"(SSB) in [10, 9]). It therefore follows that such systems cannot be 2D exponentially stable (see for example [17] for the discrete-discrete case, and [9, Chap. 4]), even in the sense of 2D exponential convergence to a non-zero equilibrium.

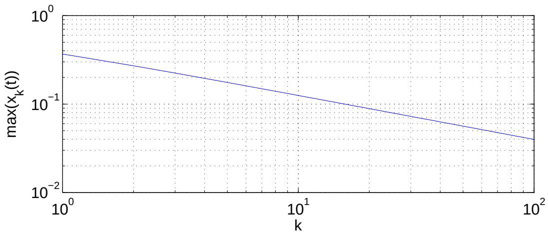

Figure 1: Local maximum vs. position within the string

1.3. Contributions

This paper gives sufficient conditions for asymptotic stability of the origin of general nonlinear 2D systems that do not require the divergence of the Lyapunov function to be strictly negative. Hence, the stability conditions are suitable to guarantee asymptotic stability of the origin of stable system, that cannot be exponentially stable, such as some classes of vehicle platoons discussed above. Further, nonlinear 2D *continuous-discrete* systems are considered. The derived stability conditions will be used to show stability and asymptotic stability of a vehicle string with variable time headway. Further, we will close a gap in the existing literature on stability of general nonlinear 2D continuous-discrete systems by providing sufficient conditions for exponential stability.

This paper is organised as follow: Section 2 clarifies the notation and mathematical preliminaries. Sections 3 and 4 deal with asymptotic stability and exponential stability of the origin of nonlinear 2D Roesser models, respectively. Illustrative examples are given in Section 5 before concluding in Section 6. Note that a shorter version on the stability of 2D nonlinear systems was published in [11].

2. Notation and Mathematical Preliminaries

Consider the nonlinear 2D continuous-discrete system (1) with the initial condition $x_{\mathrm{c}0}(k) = x_{\mathrm{c}}(0,k)$ and the meassureable boundary condition $x_{\mathrm{d}0}(t) = x_{\mathrm{d}}(t,0)$ where t is the continuous variable and k is the discrete variable. The solution $x(t,k)$ of (1) is formally given in Definition 1 in the appendix.

It will be assumed that f_{c} and f_{d} in (1) satisfy $f_{\mathrm{c}}(0,0) = 0$ and $f_{\mathrm{d}}(0,0) = 0$ and are globally Lipschitz[1].

$$\left\| \begin{pmatrix} f_{\mathrm{c}}(x_{\mathrm{c}}^a, x_{\mathrm{d}}^a) \\ f_{\mathrm{d}}(x_{\mathrm{c}}^a, x_{\mathrm{d}}^a) \end{pmatrix} - \begin{pmatrix} f_{\mathrm{c}}(x_{\mathrm{c}}^b, x_{\mathrm{d}}^b) \\ f_{\mathrm{d}}(x_{\mathrm{c}}^b, x_{\mathrm{d}}^b) \end{pmatrix} \right\| \leq K \left\| \begin{pmatrix} x_{\mathrm{c}}^a \\ x_{\mathrm{d}}^a \end{pmatrix} - \begin{pmatrix} x_{\mathrm{c}}^b \\ x_{\mathrm{d}}^b \end{pmatrix} \right\| \tag{2}$$

Throughout this paper, the notion of "positive definite", $\mathcal{K}$, and $\mathcal{K}_\infty$ functions will be used: A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite if it is continuous and satisfies

[1]Details on the Lipschitz condition and the existence and uniqueness of solutions can be found for instance in [22].

$f(0) = 0$ and $f(x) > 0$ for all $x > 0$. A function α is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is positive definite and strictly increasing. A function α is of class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it is of class $\mathcal{K}$ and in addition $\alpha(x) \to +\infty$ as $x \to +\infty$.

In [2] the authors presented the definition of a iISS-Lyapunov functions for continuous systems. A similar definition for discrete systems was given in [1]. In order to adapt these notions to 2D continuous-discrete systems, both definitions were combined to yield the notion of "2D Lyapunov function", see [11] or Definition 2 in the appendix, which will be used to show Lyapunov stability of 2D continuous-discrete systems (see Definition 6. Lyapunov stability will be shown for systems, whose initial states are bounded. To be precise, the initial conditions have to be "L_V and L_∞ bounded" (see [11] or Definition 3). This can be seen as a adaptation of L_p and L_∞ initial conditions for general nonlinear systems. Then, Lyapunov stability of 2D continuous-discrete systems can be shown:

Corollary 1 ([11] Stability of Nonlinear 2D Systems). *The nonlinear 2D system* (1) *has a stable equilibrium at the origin if there exists a 2D Lyapunov function as in Definition 2.*

The proof is based on Lemma 4 in Appendix B. The complete proof can be found in [11].

As discussed in the introduction, there exist 2D continuous-discrete systems, which are not exponentially stable (see Definition 8) but satisfy a notion of asymptotic stability, which can be found in Definition 7 in the appendix. This definition of asymptotic stability implies that the solutions of an asymptotically stable system converge to zero for $t + k \to \infty$ (not necessarily exponentially fast). This includes the cases $t \to \infty$, and $k \to \infty$ alone. Note that in the literature a different form of asymptotic stability is also defined where convergence is required for $t,k \to \infty$, e.g. [4]. To enable the proof of asymptotic stability later in the paper, the definition of 2D Lyapunov function was strengthened to "regular 2D Lyapunov functions" (see Definition 2). Further, instead of only requiring that the initial conditions are (L_V and L_∞) bounded, their derivatives have to be sufficiently smooth to allow to show asymptotic stability. See Definition 4 for details.

The proof of asymptotic stability in Section 3 uses Corollary 1 (stability of nonlinear 2D systems) above, and Corollary 5, Lemma 6 and Lemma 7 in Appendix B. Corollary 5 shows that, if a regular 2D Lyapunov function for a nonlinear 2D system exists and its initial conditions are L_V and L_∞ bounded, then the integral of V_c with respect to t and the accumulation of V_d with respect to k are bounded. Lemma 6 proves that under suitable conditions (e.g. the existence of a regular 2D Lyapunov function and L'_p and L''_∞ smooth bounded initial conditions), the states also satisfy certain smoothness conditions. Given those smoothness conditions shown in Lemma 6, a version of Barbarlat's Lemma (Lemma 7) can be derived.

Further strengthening the definition of 2D Lyapunov function yields the notation of "strict 2D Lyapunov function" (see Definition 2). It will be shown in Section 4, that if such a strict 2D Lyapunov function exists and the initial conditions are exponentially decaying (see Definition 5), the origin is exponentially stable according to Definition 8.

3. Asymptotic Stability

In this section, asymptotic stability of the origin is shown for nonlinear 2D continuous-discrete systems. In contrast to the conditions known in the literature, the sufficient conditions proposed here can be applied to systems that only allow a Lyapunov function with a nonpositive divergence — rather than a strictly negative divergence. Additional smoothness assumptions on the initial conditions and the state space system have to be made.

Theorem 2 (Asymptotic Stability of the Origin of Nonlinear 2D Systems). *Consider the nonlinear 2D system* (1) *and the related time-varying linear system* (B.7)*. If there exists a regular 2D Lyapunov function V according to Definition 2 for system* (1) *and in addition there exist two Lyapunov functions* W_c *and* W_d *and scalars* $1 \leq p < \infty$ *and* $0 < \alpha'_\mathrm{c},\alpha'_\mathrm{d},\underline{\alpha}'_\mathrm{c},\underline{\alpha}'_\mathrm{d},\overline{\alpha}'_\mathrm{c},\overline{\alpha}'_\mathrm{d}, a_\mathrm{c},a_\mathrm{d},b_\mathrm{c},b_\mathrm{d} < \infty$ *such that the initial conditions are* L'_p *and* L''_∞ *smooth bounded according to Definition 4 and equations* (B.8),(B.9) *and B.11 are satisfied for all* $t,k > 0$*, then the origin is asymptotically stable according to Definition 7.*

Proof. Note that every regular 2D Lyapunov function is also a 2D Lyapunov function and L'_p and L''_∞ smooth bounded initial conditions are by definition also L_V and L_∞ bounded initial conditions. Hence, the origin is stable by Corollary 1.

To show attractivity of the origin, consider the integral of $V_\mathrm{c}(t,k) + V_\mathrm{d}(t,k)$ along $\Omega(l) := (t,k) \in \{[0,l] \times \{\lceil l \rceil\}\} \cup \{\{l\} \times [0,\lceil l \rceil]\}$ for $l \in \mathbb{R}_+$ as:

$$U(l) := \int_0^l \left(V_\mathrm{c}(x_\mathrm{c}(t,l)) + V_\mathrm{d}(x_\mathrm{d}(t,l))\right) \mathrm{d}t + \sum_0^l \left(V_\mathrm{c}(x_\mathrm{c}(l,k)) + V_\mathrm{d}(x_\mathrm{d}(l,k))\right).$$

There exists a ϕ such that $U(l) \leq \phi$ for all l due to the results in Lemma 4 and Corollary 5 in Appendix B. Since the first derivatives of $x(t,k)$ with respect to t and the first differences with respect to k are L_∞ bounded (see Lemma 6 in Appendix B) and since the Lyapunov function components V_c and V_d are differentiable as stated in Definition 2, define for $i \in \{c,d\}$

$$d_{i\mathrm{c}}(l) := \sup_{0 \leq t \leq l} \left|\dot{V}_i(x_i(t,l))\right|, \quad d_{i\mathrm{d}}(l) := \sup_{0 \leq k \leq l} \left|\Delta V_i(x_i(l,k))\right|. \tag{3}$$

Note that for the derivatives with respect to t the above bounds follow immediately from the results in Lemma 6 and the chain rule of differentiation. For the differences with respect to k, similar arguments can be made. For details see [9, Sec. 5.5].

Again using the 2D version of Barbalat's Lemma (see Lemma 7 in Appendix B), we can conclude that the first derivatives and differences tend to zero as $t,k \to \infty$ and are uniformly convergent in both directions. That allows to interchange the order of supremum and limit and thus to conclude $\lim_{l\to\infty} d_\mathrm{cc}(l) \leq \sup_{t\geq 0} \lim_{l\to\infty} \left|\dot{V}_\mathrm{c}(x_\mathrm{c}(t,l))\right| = 0$. The same argument can be made to show that $d_\mathrm{cd}(l)$, $d_\mathrm{dc}(l)$ and $d_\mathrm{dd}(l)$ tend to zero.

Define the maximum of $V_\mathrm{c}(t,k)$ along $\Omega(l)$ by $\overline{V}_\mathrm{c}(l) := \max_{(t,k)\in\Omega(l)} V_\mathrm{c}(t,k)$. To find a lower bound on $U(l)$, note that if $\overline{V}_\mathrm{c}(l)$ occurs along the part of $\Omega(l)$ where $(t,k) \in [0,l] \times \{\lceil l \rceil\}$, we can bound the integral of $V_\mathrm{c}(t,k)$ over $\Omega(l)$ from below by the area of a triangle. The height of the triangle is given by $\overline{V}_\mathrm{c}(l)$. Given that $V_\mathrm{c}(t,l)$ can decay at most with slope $-d_\mathrm{cc}(l)$, the base of the smallest possible triangle is equal to $\overline{V}_\mathrm{c}(l)\Big/d_\mathrm{cc}(l)$

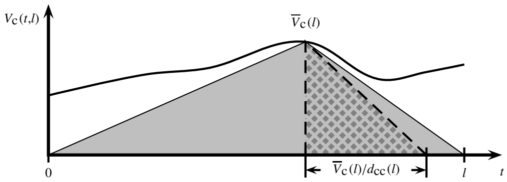

Figure 2: Illustration of lower bound of $U(l)$ using a triangle

(if $\overline{V}_{\mathrm{c}}(l)$ occurs at the integration boundary) or the length of the integration interval l, see also Figure 2.

In case $\overline{V}_{\mathrm{c}}(l)$ occurs at $(t,k) \in \{\lceil l \rceil\} \times [0, l]$ a similar argument can be followed. Details can be found in [9, Sec. 4.5 and 5.5]. Thus, following similar arguments for the component depending on V_{d}, $U(l)$ can be bounded from below by

$$U(l) \geq \min\left\{\frac{\overline{V}_{\mathrm{c}}^2(l)}{2d_{\mathrm{cc}}(l)}, \frac{\overline{V}_{\mathrm{c}}^2(l)}{2d_{\mathrm{cd}}(l)}, \frac{\overline{V}_{\mathrm{c}}(l)l}{2}\right\} + \min\left\{\frac{\overline{V}_{\mathrm{d}}^2(l)}{2d_{\mathrm{dc}}(l)}, \frac{\overline{V}_{\mathrm{d}}^2(l)}{2d_{\mathrm{dd}}(l)}, \frac{\overline{V}_{\mathrm{d}}(l)l}{2}\right\}.$$

Since $\overline{V}_{\mathrm{c}}(l) \leq V_{\mathrm{c}}(M_{\mathrm{c}})$ and $\overline{V}_{\mathrm{d}}(l) \leq V_{\mathrm{d}}(M_{\mathrm{d}})$ where $\overline{V}_{\mathrm{c}}(l)$ and $\overline{V}_{\mathrm{d}}(l)$ are the maximum of V_{c} and V_{d} in the region $\Omega(l)$, this implies

$$\overline{V}_{\mathrm{c}}^2(l) \leq 2\phi \cdot \max\left\{d_{\mathrm{cc}}(l), \quad d_{\mathrm{cd}}(l), \quad \frac{V_{\mathrm{c}}(M_{\mathrm{c}})}{l}\right\}. \tag{4}$$

Note that as $l \to \infty$ each component of the maximum in (4) goes to 0. Thus, it follows that $\lim_{t+k\to\infty} |V_{\mathrm{c}}(x_{\mathrm{c}}((t,k))| = 0$ and $\lim_{t+k\to\infty} |x_{\mathrm{c}}(t,k)| = 0$. The convergence of x_d can be shown following similar steps as for the convergence of x_c. More details can be found in [9].

It should be noted that the assumptions used to guarantee asymptotic stability of the origin seem to be rather restrictive. However, requiring smoothness of the initial conditions, the state space description and the Lyapunov functions allows us to formulate sufficient conditions for asymptotic stability even if the divergence, div$V$, is negative *semi-definite*. It should also be noted again that according to the definition of asymptotic stability used here, the conditions in Theorem 2 guarantee the stronger convergence to zero under $t + k \to \infty$ in contrast to only $t,k \to \infty$.

4. Exponential Stability

The previous section presented sufficient conditions to guarantee asymptotic stability of general nonlinear 2D continuous-discrete Roesser models. It is important to notice that these conditions do not rely on the divergence of the Lyapunov function to be strictly negative. This is a major advantage when discussing asymptotic stability of models, that can never admit a Lyapunov function with strictly negative divergence, such as 2D descriptions of vehicle platoons. However, the convergence to the asymptotically stable origin might be very slow. In contrast, if a strict 2D Lyapunov function with a strictly negative divergence exists, the result above can be strengthened and exponential stability can be guaranteed:

Theorem 3 (Exponential Stability of Nonlinear 2D Systems). *The nonlinear 2D system* (1) *is exponentially stable, if there exist a strict 2D Lyapunov function as defined in Definition 2 and positive $p_{\mathrm{c}}, p_{\mathrm{d}} < \infty$ such that $\underline{\alpha}_{\mathrm{c}}(|x_{\mathrm{c}}|) \geq |x_{\mathrm{c}}|^{p_{\mathrm{c}}}$ and $\underline{\alpha}_{\mathrm{d}}(|x_{\mathrm{d}}|) \geq |x_{\mathrm{d}}|^{p_{\mathrm{d}}}$.*

Proof. First, consider the Lyapunov function candidate

$$\tilde{V} = \begin{pmatrix} \tilde{V}_{\mathrm{c}}(x_{\mathrm{c}}) \\ \tilde{V}_{\mathrm{d}}(x_{\mathrm{d}}) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} V_{\mathrm{c}} \\ \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k-1} V_{\mathrm{d}} \end{pmatrix} \tag{5}$$

with positive constants η_{c} and $\eta_{\mathrm{d}} > 1$. It will be shown that when choosing η_{c} and η_{d} sufficiently small, $\tilde{V}$ is a 2D Lyapunov function according to Definition 2. This will imply that an exponentially scaled version of the 2D system (1) is stable by Corollary 1, which will show that the original system must be exponentially stable.

Note that

$$\begin{aligned} \dot{\tilde{V}}_{\mathrm{c}} &= \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} \dot{V}_{\mathrm{c}} + \eta_{\mathrm{c}} \tilde{V}_{\mathrm{c}} && (6) \\ \Delta \tilde{V}_{\mathrm{d}} &= \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} \Delta V_{\mathrm{d}} + (\eta_{\mathrm{d}} - 1) \tilde{V}_{\mathrm{d}}. && (7) \end{aligned}$$

Applying condition (A.7) of the definition for a strict 2D Lyapunov function yields

$$\begin{aligned} \dot{\tilde{V}}_{\mathrm{c}} &\leq - \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} a_{\mathrm{c}} V_{\mathrm{c}} + \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} b_{\mathrm{c}} V_{\mathrm{d}} + \eta_{\mathrm{c}} \tilde{V}_{\mathrm{c}} \\ &\leq (\eta_{\mathrm{c}} - a_{\mathrm{c}}) \tilde{V}_{\mathrm{c}} + b_{\mathrm{c}} \eta_{\mathrm{d}} \tilde{V}_{\mathrm{d}} && (8) \\ \Delta \tilde{V}_{\mathrm{d}} &\leq - \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} a_{\mathrm{d}} V_{\mathrm{d}} + \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} b_{\mathrm{d}} V_{\mathrm{c}} + (\eta_{\mathrm{d}} - 1) \tilde{V}_{\mathrm{d}} \\ &\leq (\eta_{\mathrm{d}} - 1 - a_{\mathrm{d}} \eta_{\mathrm{d}}) \tilde{V}_{\mathrm{d}} + b_{\mathrm{d}} \tilde{V}_{\mathrm{c}} && (9) \end{aligned}$$

Choosing $\eta_{\mathrm{c}} < a_{\mathrm{c}}$ and $\eta_{\mathrm{d}} < \frac{1}{1-a_{\mathrm{d}}}$ guarantees that $\tilde{V}_{\mathrm{c}}$ and $\tilde{V}_{\mathrm{d}}$ satisfy conditions (A.2) and (A.3), respectively. Consider now the divergence and condition (A.9):

$$\begin{aligned} \operatorname{div} \tilde{V} &= \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} \operatorname{div} V + \eta_{\mathrm{c}} \tilde{V}_{\mathrm{c}} + (\eta_{\mathrm{d}} - 1) \tilde{V}_{\mathrm{d}} \\ &\leq \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k} (-\gamma_{\mathrm{c}} V_{\mathrm{c}} - \gamma_{\mathrm{d}} V_{\mathrm{d}}) + \eta_{\mathrm{c}} \tilde{V}_{\mathrm{c}} + (\eta_{\mathrm{d}} - 1) \tilde{V}_{\mathrm{d}} \\ &= (\eta_{\mathrm{c}} - \gamma_{\mathrm{c}}) \tilde{V}_{\mathrm{c}} + (\eta_{\mathrm{d}} - 1 - \eta_{\mathrm{d}} \gamma_{\mathrm{d}}) \tilde{V}_{\mathrm{d}}. && (10) \end{aligned}$$

If $\eta_{\mathrm{c}} < \gamma_{\mathrm{c}}$ and $\eta_{\mathrm{d}} < \frac{1}{1-\gamma_{\mathrm{d}}}$, $\tilde{V}$ satisfies (A.4). Note that the initial conditions of V decay exponentially and thus $\tilde{V}_{\mathrm{c}}(x_{\mathrm{c}0})$ and $\tilde{V}_{\mathrm{d}}(x_{\mathrm{d}0})$ are bounded by

$$\tilde{V}_{\mathrm{c}}(x_{\mathrm{c}0}(k)) = \eta_{\mathrm{d}}^{k} V_{\mathrm{c}}(x_{\mathrm{c}0}(k)) \leq (\eta_{\mathrm{d}} \mu_{\mathrm{d}})^{k} \kappa_{\mathrm{c}} \tag{11}$$

$$\tilde{V}_{\mathrm{d}}(x_{\mathrm{d}0}(t)) = \mathrm{e}^{\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{-1} V_{\mathrm{d}}(x_{\mathrm{d}0}(t)) \leq \mathrm{e}^{(\eta_{\mathrm{c}} - \mu_{\mathrm{c}}) t} \frac{\kappa_{\mathrm{d}}}{\eta_{\mathrm{d}}}. \tag{12}$$

Choosing $\eta_{\mathrm{c}} < \mu_{\mathrm{c}}$ and $\eta_{\mathrm{d}} < \frac{1}{\mu_{\mathrm{d}}}$ also guarantees that the initial conditions of $\tilde{V}$ are in L_∞

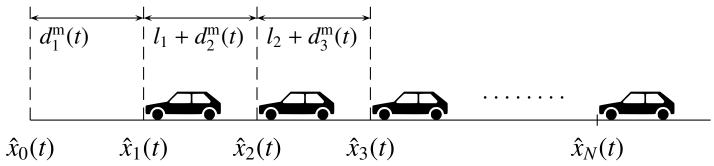

Figure 3: Platoon of N vehicles

and L_V. Hence, choosing

$$\eta_{\mathrm{c}} < \min\{a_{\mathrm{c}}, \quad \gamma_{\mathrm{c}}, \quad \mu_{\mathrm{c}}\}, \quad \text{and} \tag{13}$$

$$\eta_{\mathrm{d}} < \min\left\{\frac{1}{1-a_{\mathrm{d}}}, \quad \frac{1}{1-\gamma_{\mathrm{d}}}, \quad \frac{1}{\mu_{\mathrm{d}}}\right\} \tag{14}$$

allows us to apply Corollary 1, which guarantees that there exists a $C < \infty$ such that $\tilde{V}_{\mathrm{c}}, \tilde{V}_{\mathrm{d}} \le C$. Since $|x_{\mathrm{c}}|^{p_{\mathrm{c}}} \le \underline{\alpha}_{\mathrm{c}}(|x_{\mathrm{c}}|) \le V_{\mathrm{c}}(x_{\mathrm{c}})$ and $|x_{\mathrm{d}}|^{p_{\mathrm{d}}} \le \underline{\alpha}_{\mathrm{d}}(|x_{\mathrm{d}}|) \le V_{\mathrm{d}}(x_{\mathrm{d}})$ we can conclude that

$$|x_{\mathrm{c}}| \le \mathrm{e}^{-\frac{\eta_{\mathrm{c}}}{p_{\mathrm{c}}}t}\eta_{\mathrm{d}}^{-\frac{k}{p_{\mathrm{c}}}}C^{1/p_{\mathrm{c}}} \tag{15}$$

$$|x_{\mathrm{d}}| \le \mathrm{e}^{-\frac{\eta_{\mathrm{c}}}{p_{\mathrm{d}}}t}\eta_{\mathrm{d}}^{-\frac{k-1}{p_{\mathrm{d}}}}C^{1/p_{\mathrm{d}}}. \tag{16}$$

Thus, the system is exponentially stable according to Definition 8.

Note that the rates with which $|x_{\mathrm{c}}|$ and $|x_{\mathrm{d}}|$ decay depend on the Lyapunov functions V_{c} and V_{d}, and constants p_{c} and p_{d}.

5. Example

We discuss stability and asymptotic stability of a nonlinear vehicle string with a variable time headway. (This is a nonlinear extension of [10, Example 1] where a fixed time headway is used.)

System Description. The 2D system studied in this example is used to model a simple vehicle platoon as depicted in Figure 3 where the absolute position of vehicle k and its velocity are denoted $\hat{x}_k(t)$ and $\hat{v}_k(t)$, respectively.

The leading vehicle is driven by an automatic controller which aims to follow a given reference signal. Each other vehicle monitors the distance towards the preceding vehicle's rear bumper bar using radar or lidar. Hence, the distance between the neighbouring vehicles, that is $x_{k-1}(t) - x_k(t)$, is described by the measured distance towards the read of the preceding vehicle $d_k^{\mathrm{m}}(t)$ and the length of the preceding vehicle l_{k-1}. Each vehicle (apart from the leading vehicle) is driven by an automatic controller, which aims to maintain the prescribed distance $d_k^{\mathrm{k}} + h_k(t)\hat{v}_k(t)$ between vehicles k and $k-1$, where d_k^{s} is the static safety distance. In order to guarantee string stability of the platoon, it is desirable that the distances between the vehicles (ignoring the static

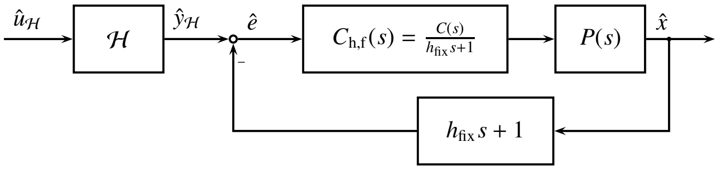

Figure 4: Block diagram of subsystem with variable time headway

distance) is proportional to the vehicles' speed leading to the introduction of the time headway $h_k(t)$. (It is known that string stability can be achieved if the time headway is chosen as a constant greater than the 'critical' time headway h_0, [10, Example 1].) In its simplest form, the time headway is constant and uniform, that is h in seconds, ensuring that in steady state each vehicle follows h seconds behind its predecessor. However, in case of high vehicle speed, the steady state distances grow considerably. Here, the variable time headway $h_k(t) = h_{\mathrm{var}}$ (which depends on the states of vehicle k) is used. Then, when ignoring the static terms l_{k-1} and d_k^{s}, the local position error, which vehicle k aims to minimise, is given by $\hat{e}_k(t) = x_{k-1}(t) - x_k(t) - h_k(t)\hat{v}_k(t)$. The form of h_{var} considered here (given in (20)) was proposed in [24] (including an additional upper saturation bound). Yet, string stability of the system has not been shown analytically but through simulations.

Consider the plant model $P(s) = \frac{1}{s^2+2\eta v_0 s}$ and the local PID controller $C(s) = k_{\mathrm{p}} + \frac{k_{\mathrm{i}}}{s} + \frac{k_{\mathrm{d}}s}{Ts+1}$. The variable time headway can be written as a sum of a fixed part h_{fix} and variable part $\Delta h_{\mathrm{var}}(x)$ depending on the state x, i. e. $h_{\mathrm{var}} = h_{\mathrm{fix}} + \Delta h_{\mathrm{var}}(x)$. Note that h_{fix} is a constant greater than the critical time headway of $h_0 = 1.18$ for this particular setting. A pole at $-\frac{1}{h_{\mathrm{fix}}}$ is added to each local controller.

Stability. The stability of the system can be analysed by transforming it into the scheme with the abstract block $\mathcal{H}$ in Fig. 4, where the position of the kth vehicle is the input for $\mathcal{H}$ of subsystem $k+1$, i. e. $\hat{x}(t,k) = \hat{u}_{\mathcal{H}}(t,k+1)$. We will use the following state space description for the additional state $x_{1_2}(t)$ of the system $\mathcal{H}$:

$$\dot{x}_{1_2}(t) = -\frac{1}{h_{\mathrm{var}}}x_{1_2}(t) + \frac{\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}}{h_{\mathrm{var}}}u_{\mathrm{H}}(t), \tag{17}$$

$$y_{\mathrm{H}}(t) = \frac{\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}}{h_{\mathrm{var}}}x_{1_2}(t) + \frac{h_{\mathrm{fix}}}{h_{\mathrm{var}}}u_{\mathrm{H}}(t). \tag{18}$$

Note that with h_{var} fixed, the frozen system $\mathcal{H}$ is linear, time invariant with transfer function $H(s) = \frac{h_{\mathrm{fix}}s+1}{h_{\mathrm{var}}s+1}$. In general, we allow h_{var} to be any time varying function that satisfies $h_{\mathrm{var}} \geq h_{\mathrm{fix}}$.

The vehicle string can now be modelled as a 2D system where the first independent variable is continuous time t, the second is the discrete position in the string k. Thus, the system is described by

$$\begin{pmatrix} \dot{x}_{\mathrm{c}_1}(t,k) \\ \dot{x}_{\mathrm{c}_2}(t,k) \\ \Delta x_{\mathrm{d}}(t,k) \end{pmatrix} = \underbrace{\begin{bmatrix} A_0 & b_0\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & b_0 h_{\mathrm{fix}}/h_{\mathrm{var}} \\ 0 & -1/h_{\mathrm{var}} & \sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} \\ c & 0 & -1 \end{bmatrix}}_{A(t)} \begin{pmatrix} x_{\mathrm{c}_1}(t,k) \\ x_{\mathrm{c}_2}(t,k) \\ x_{\mathrm{d}}(t,k) \end{pmatrix}$$

where $x_{\mathrm{c}_1}(t,k)$ are the existing states of the controller and the vehicle model and, therefore, there exists a state space realisation with

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -2\eta v_0 & 1 & 0 & 0 \\ -\frac{1}{h_{\mathrm{fix}}}\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & -\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & -\frac{1}{h_{\mathrm{fix}}} & \frac{1}{h_{\mathrm{fix}}} & -\frac{k_{\mathrm{d}}}{h_{\mathrm{fix}}T^2} \\ -k_{\mathrm{i}} & -h_{\mathrm{fix}}k_{\mathrm{i}} & 0 & 0 & 0 \\ -1 & -h_{\mathrm{fix}} & 0 & 0 & -\frac{1}{T} \end{bmatrix}, \tag{19}$$

$b_0 = \begin{pmatrix} 0 & 0 & \frac{1}{h_{\mathrm{fix}}}\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & k_{\mathrm{i}} & 1 \end{pmatrix}^{\mathrm{T}}$ and $c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. Note that the eigenvalues of $A_{\mathrm{cc}} = \begin{bmatrix} A_0 & b_0\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} \\ 0 & -1/h_{\mathrm{var}} \end{bmatrix}$ have negative real parts for $h_{\mathrm{fix}}, h_{\mathrm{var}} > 0$.

Consider the Lyapunov function V with $V_{\mathrm{c}}(x_{\mathrm{c}}) = x_{\mathrm{c}_1}^{\mathrm{T}}(t,k)Px_{\mathrm{c}_1}(t,k) + x_{\mathrm{c}_2}^2$ and $V_{\mathrm{d}}(x_{\mathrm{d}}) = x_{\mathrm{d}}^{\mathrm{T}}(t,k)x_{\mathrm{d}}(t,k)$ such that $\operatorname{div} V = x^{\mathrm{T}}Qx$ with

$$Q = \begin{bmatrix} A_0^{\mathrm{T}}P + PA_0 + c^{\mathrm{T}}c & Pb_0\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & Pb_0 h_{\mathrm{fix}}/h_{\mathrm{var}} \\ b_0^{\mathrm{T}}P\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & -2/h_{\mathrm{var}} & \sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} \\ b_0^{\mathrm{T}}Ph_{\mathrm{fix}}/h_{\mathrm{var}} & \sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & -1 \end{bmatrix}.$$

Applying the Schur complement twice, the requirement $Q \leq 0$ is equivalent to $A_0^{\mathrm{T}}P + PA_0 + c^{\mathrm{T}}c + Pb_0b_0^{\mathrm{T}}P \leq 0$. Applying the Bounded Real Lemma we can show that this is equivalent to the condition $\left\|c\,(j\omega I - A_0)^{-1}\,b_0\right\|_\infty \leq 1$. $\Gamma_0(j\omega) = c\,(j\omega I - A_0)^{-1}\,b_0$ is the transfer function from the kth to the $k+1$th vehicle for $h_{\mathrm{var}} = h_{\mathrm{fix}}$. Since the time headway h_{fix} is greater than the infimal time headway $h_0 = 1.18$, $|\Gamma(j\omega)| \leq 1$ for all ω and $|\Gamma(j\omega)| < 1$ for $\omega \neq 0$. Thus, a positive definite Matrix P exists such that Q is negative semi-definite independently of h_{var} (for $h_{\mathrm{var}} > h_0$) and the origin is stable.

Asymptotic Stability. Consider the variable time headway

$$h_{\mathrm{var}}(t,k) = \begin{cases} h_{\mathrm{ss}} + k_{\mathrm{h}}\,(\hat{v}(t,k) - \hat{v}(t,k-1)) & \text{for } h_{\mathrm{min}} \leq h_{\mathrm{var}}(t,k), \\ h_{\mathrm{min}} & \text{else,} \end{cases} \tag{20}$$

where the time headway in steady state is $h_{\mathrm{ss}} = 1.4$, $k_{\mathrm{h}} = 0.05$ and the variable time headway is saturated at $h_{\mathrm{fix}} = h_{\mathrm{min}} = 1.2$. The motivation for the choice (20) is that in case the vehicle is driving slower than its predecessor, the variable time headway decreases and the vehicle thus accelerates faster and reaches its desired position faster.

To ensure asymptotic stability, we seek two Lyapunov functions W_{c} and W_{d} satis-

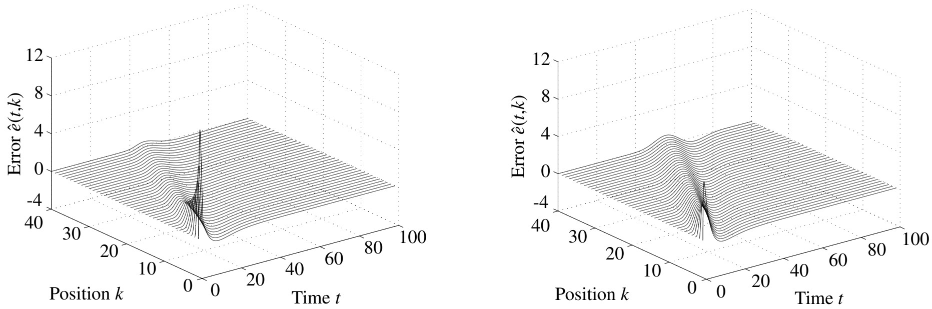

Figure 5: Local error for a string with variable time headway (left) and constant time headway of $h = 1.4$ (right)

fying (B.8)-(B.9). The Jacobian matrix is given by

$$F = A(t) + k_\mathrm{h} \begin{bmatrix} b_0\left(\frac{h_\mathrm{fix}-h_\mathrm{var}/2}{\sqrt{h_\mathrm{var}-h_\mathrm{fix}}h_\mathrm{var}^2}\hat{v}(t,k) - \frac{h_\mathrm{fix}}{h_\mathrm{var}^2}\hat{v}(t,k-1)\right) \\ \left(\frac{\hat{v}(t,k)}{h_\mathrm{var}^2} + \frac{h_\mathrm{fix}-h_\mathrm{var}/2}{\sqrt{h_\mathrm{var}-h_\mathrm{fix}}h_\mathrm{var}^2}\hat{v}(t,k-1)\right) \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & \cdots \end{bmatrix}.$$

As $F_\mathrm{dd} = A_\mathrm{dd}(t) = -1$ it is straight forward to find a suitable function W_d. Note also that the difference between F_cc and $A_\mathrm{cc}(t)$ is a perturbation of rank 1. As $A_\mathrm{cc}(t)$ is strictly Hurwitz for $h_\mathrm{fix}, h_\mathrm{var} > 0$ it is always possible to choose a sufficiently small k_h such that F_cc is strictly Hurwitz for a given range of velocities to ensure a suitable Lyapunov function W_c exists. To ensure condition B.11 is satisfied, note that according to [3, Fact 6.4.20] $\mathrm{rank}Q = \mathrm{rank}(1) + \mathrm{rank}\left(\frac{h_\mathrm{var}+h_\mathrm{fix}}{h_\mathrm{var}}\right) + \mathrm{rank}\left(A_0^\mathrm{T}P + PA_0 + c^\mathrm{T}c + Pb_0b_0^\mathrm{T}P\right)$. There exists a P such that the last term drops one rank. Thus, Q drops one rank and there exists a positive definite matrix R such that $Q = -A^\mathrm{T}RA$. Post and pre multiplying this with x yields B.11. Hence the origin is asymptotically stable.

Simulations. A string of forty vehicles has been simulated. The local error is shown on the left of Fig. 5. When comparing these results to the simulation with a constant time headway of $h = 1.4$ (see the right of Fig. 5) one observes that with a variable time headway with $h_\mathrm{ss} = 1.4$ the error for the first vehicle increases to a maximal value that is twice as high as in case of the constant time headway $h = 1.4$. This is due to the decreased time headway, and consequently the desired distance between the first vehicle and reference position decreases temporarily and the error increases. However, with the variable time headway, the local errors decrease more rapidly than choosing a constant time headway.

6. Conclusion

Sufficient conditions guaranteeing Lyapunov stability, asymptotic stability and exponential stability of nonlinear 2D systems have been presented based on the theory of integral input-to-state stability. Since for guaranteeing Lyapunov stability and asymptotic stability the divergence of the 2D Lyapunov function is only required to be nonpositive, additional assumptions have to be made. To guarantee stability of nonlinear 2D continuous-discrete systems, the only additional assumption is that the iISS-Lyapunov function derivative $\dot{V}_{\mathrm{c}}$ and the difference ΔV_{d} depend on the Lyapunov function in a certain form. However, the proof for asymptotic stability also requires certain smoothness conditions on the initial conditions, the state space equations and the Lyapunov function. In some ways this had to be expected as it was noted in [25, Remark 3] in order to show global asymptotic stability for nonpositive differences, at least the assumptions on the initial conditions need to be stronger than merely boundedness. When strengthening the 2D Lyapunov function by requiring a strictly negative divergence, the smoothness conditions of the initial conditions can be relaxed and exponential stability can be guaranteed.

AppendixA. Definitions

Definition 1. $\{x_{\mathrm{c}}(\cdot,\cdot), x_{\mathrm{d}}(\cdot,\cdot)\} : (\mathbb{R}^+,\mathbb{N}) \to (\mathbb{R}^{n_c},\mathbb{R}^{n_d})$ *is a solution of* (1) *for* $t \in [0,\infty)$ *and* $k \in \{1,2,\dots\}$ *if* $x_{\mathrm{c}}(t,k)$ *is differentiable with respect to* t *everywhere and equation* (1) *holds for all* t *and* k.

Definition 2. *A 2D function* $V^{\mathrm{T}} = \begin{pmatrix} V_{\mathrm{c}}(x_{\mathrm{c}}) & V_{\mathrm{d}}(x_{\mathrm{d}}) \end{pmatrix}$ *is called a 2D Lyapunov function for system* (1) *if* $V_{\mathrm{c}}(x_{\mathrm{c}})$ *is a continuous-time iISS-Lyapunov function (which is continuously differentiable with respect to* x_{c} *as in [2]) for subsystem* $\dot{x}_{\mathrm{c}}(t,k) = f_{\mathrm{c}}(x_{\mathrm{c}}(t,k),x_{\mathrm{d}}(t,k))$ *and* $V_{\mathrm{d}}(x_{\mathrm{d}})$ *is a discrete-time iISS-Lyapunov function (which is continuous in* x_{d} *as in [1]) for subsystem* $\Delta x_{\mathrm{d}}(t,k) = f_{\mathrm{d}}(x_{\mathrm{c}}(t,k),x_{\mathrm{d}}(t,k))$*, that is there exist functions* $\overline{\alpha}_{\mathrm{c}}, \underline{\alpha}_{\mathrm{c}}, \overline{\alpha}_{\mathrm{d}}, \underline{\alpha}_{\mathrm{d}} \in \mathcal{K}_\infty$*, positive definite functions* $\alpha_{\mathrm{c}}, \alpha_{\mathrm{d}}$ *and constants* $0 \le b_{\mathrm{c}}, b_{\mathrm{d}} < \infty$ *such that*

$$\underline{\alpha}_{\mathrm{c}}(|x_{\mathrm{c}}|) \le V_{\mathrm{c}}(x_{\mathrm{c}}) \le \overline{\alpha}_{\mathrm{c}}(|x_{\mathrm{c}}|), \quad \underline{\alpha}_{\mathrm{d}}(|x_{\mathrm{d}}|) \le V_{\mathrm{d}}(x_{\mathrm{d}}) \le \overline{\alpha}_{\mathrm{d}}(|x_{\mathrm{d}}|), \tag{A.1}$$
$$\dot{V}_{\mathrm{c}}(x_{\mathrm{c}}) \le -\alpha_{\mathrm{c}}(V_{\mathrm{c}}(x_{\mathrm{c}})) + b_{\mathrm{c}} V_{\mathrm{d}}(x_{\mathrm{d}}), \tag{A.2}$$
$$\Delta V_{\mathrm{d}}(x_{\mathrm{d}}) \le -\alpha_{\mathrm{d}}(V_{\mathrm{d}}(x_{\mathrm{d}})) + b_{\mathrm{d}} V_{\mathrm{c}}(x_{\mathrm{c}}), \quad \textit{and} \tag{A.3}$$
$$\mathrm{div} V = \dot{V}_{\mathrm{c}}(x_{\mathrm{c}}) + \Delta V_{\mathrm{d}}(x_{\mathrm{d}}) \le 0 \quad \forall x_{\mathrm{c}}, x_{\mathrm{d}}. \tag{A.4}$$

A 2D Lyapunov function V *is called a regular 2D Lyapunov function for system* (1) *if it satisfies* (A.1) *and* (A.4)*,* V_{d} *is continuously differentiable with respect to* x_{d} *and there exist constants* $0 \le a_{\mathrm{c}} < \infty$ *and* $0 \le a_{\mathrm{d}} < 2$ *such that*

$$\dot{V}_{\mathrm{c}}(x_{\mathrm{c}}) \le -a_{\mathrm{c}} V_{\mathrm{c}}(x_{\mathrm{c}}) + b_{\mathrm{c}} V_{\mathrm{d}}(x_{\mathrm{d}}), \quad \textit{and} \tag{A.5}$$
$$\Delta V_{\mathrm{d}}(x_{\mathrm{d}}) \le -a_{\mathrm{d}} V_{\mathrm{d}}(x_{\mathrm{d}}) + b_{\mathrm{d}} V_{\mathrm{c}}(x_{\mathrm{c}}) \quad \forall x_{\mathrm{c}}, x_{\mathrm{d}}. \tag{A.6}$$

A 2D Lyapunov function V *is called a strict 2D Lyapunov function for system* (1) *if there exist positive constants* $a_{\mathrm{c}}, b_{\mathrm{c}}, b_{\mathrm{d}}, \gamma_{\mathrm{c}} < \infty$ *and* $a_{\mathrm{d}}, \gamma_{\mathrm{d}} < 1$ *such that*

$$\dot{V}_{\mathrm{c}}(x_{\mathrm{c}}) \le -a_{\mathrm{c}} V_{\mathrm{c}}(x_{\mathrm{c}}) + b_{\mathrm{c}} V_{\mathrm{d}}(x_{\mathrm{d}}), \tag{A.7}$$
$$\Delta V_{\mathrm{d}}(x_{\mathrm{d}}) \le -a_{\mathrm{d}} V_{\mathrm{d}}(x_{\mathrm{d}}) + b_{\mathrm{d}} V_{\mathrm{c}}(x_{\mathrm{c}}), \quad \textit{and} \tag{A.8}$$
$$\mathrm{div} V \le -\gamma_{\mathrm{c}} V_{\mathrm{c}}(x_{\mathrm{c}}) - \gamma_{\mathrm{d}} V_{\mathrm{d}}(x_{\mathrm{d}}) \quad \forall x_{\mathrm{c}}, x_{\mathrm{d}}. \tag{A.9}$$

Note that the main difference between the 2D Lyapunov function and the regular 2D Lyapunov function above is that the first term on the right hand side of (A.5) is $a_{\rm c}V_{\rm c}(x_{\rm c})$ instead of the more general form $\alpha_{\rm c}(V_{\rm c}(x_{\rm c}))$ in (A.2). Further, component $V_{\rm d}$ of a regular 2D Lyapunov function must be differentiable with respect to $x_{\rm d}$ rather than just continuous as in the case of 2D Lyapunov functions. The main difference between the above definitions and the definitions for iISS-Lyapunov functions in [2] and [1] are that the last terms in (A.2), (A.3), (A.5) and (A.6) explicitly contain $V_{\rm c}(x_{\rm c})$ and $V_{\rm d}(x_{\rm d})$ instead of general class $\mathcal{K}_\infty$ functions.

Definition 3 (L_V and L_∞ Bounded Initial Conditions, [11]). *Given positive definite functions $V_{\rm c}$ and $V_{\rm d}$, the initial conditions of the nonlinear two-dimensional system* (1) *are L_V and L_∞ bounded, if there exist $\xi_{\rm c}, \xi_{\rm d}, \zeta_{\rm c}, \zeta_{\rm d} < \infty$ such that*

$$\|x_{\rm c0}(\cdot)\|_V := \sum_{k=0}^{\infty} V_{\rm c}\left(x_{\rm c0}(k)\right) \le \xi_{\rm c}, \quad \|x_{\rm d0}(\cdot)\|_V := \int_0^\infty V_{\rm d}\left(x_{\rm d0}(t)\right) {\rm d}t \le \xi_{\rm d},$$

$$\|x_{\rm c0}(\cdot)\|_\infty = \sup_{k>0} |x_{\rm c0}(k)| \le \zeta_{\rm c} \quad \textit{and} \quad \|x_{\rm d0}(\cdot)\|_\infty = \sup_{t\ge 0} |x_{\rm d0}(t)| \le \zeta_{\rm d}.$$

Definition 4 (L'_p and L''_∞ Smooth Bounded Initial Conditions (SBIC)). *Given positive definite functions $V_{\rm c}$ and $V_{\rm d}$ and an integer $1 \le p < \infty$, the initial conditions of the nonlinear 2D system* (1) *are smoothly bounded if they are L_V and L_∞ bounded according to Definition 3 and in addition there exist $\xi'_{\rm c}, \xi'_{\rm d}, \zeta'_{\rm c}, \zeta'_{\rm d}, \zeta''_{\rm c}, \zeta''_{\rm d} < \infty$ such that*

$$\|\Delta x_{\rm c0}(\cdot)\|_p^p = \sum_{k=0}^{\infty} |\Delta x_{\rm c0}(k)|^p \le \xi'_{\rm c}, \quad \|\dot{x}_{\rm d0}(\cdot)\|_p^p = \int_0^\infty |\dot{x}_{\rm d0}(t)|^p \le \xi'_{\rm d},$$

$$\|\Delta x_{\rm c0}(\cdot)\|_\infty = \sup_{k\ge 0} |\Delta x_{\rm c0}(k)| \le \zeta'_{\rm c}, \quad \|\dot{x}_{\rm d0}(\cdot)\|_\infty = \sup_{t\ge 0} |\dot{x}_{\rm d0}(t)| \le \zeta'_{\rm d},$$

$$\|\Delta^2 x_{\rm c0}(\cdot)\|_\infty = \sup_{k\ge 0} |\Delta^2 x_{\rm c0}(k)| \le \zeta''_{\rm c}, \quad \|\ddot{x}_{\rm d0}(\cdot)\|_\infty = \sup_{t\ge 0} |\ddot{x}_{\rm d0}(t)| \le \zeta''_{\rm d}.$$

Definition 5 (Exponentially Decaying Initial Conditions). *Given positive definite functions $V_{\rm c}$, $V_{\rm d}$, the initial conditions of the nonlinear 2D system* (1) *are exponentially decaying, if there exist positive constants $\mu_{\rm c}$ and $\mu_{\rm d} < 1$ and $\kappa_{\rm c}, \kappa_{\rm d} < \infty$ such that $V_{\rm c}(x_{\rm c0}(k)) \le \kappa_{\rm c}\mu_{\rm d}^k$ and $V_{\rm d}(x_{\rm d0}(t)) \le \kappa_{\rm d}{\rm e}^{-\mu_{\rm c}t}$.*

Definition 6 (Stability of Nonlinear 2D Systems). *Consider the autonomous nonlinear 2D system* (1)*. The origin is globally Lyapunov stable if for each $M > 0$ there exist $\xi_{\rm c}(M), \xi_{\rm d}(M), \zeta_{\rm c}(M), \zeta_{\rm d}(M) > 0$ such that if the initial conditions are L_V and L_∞ bounded with bounds $\xi_{\rm c}(M)$,$\xi_{\rm d}(M)\zeta_{\rm c}(M)$,$\zeta_{\rm d}(M)$, then all solutions satisfy $|x(t,k)| \le M$ for all $t,k > 0$.*

Definition 7 (Asymptotic Stability of Nonlinear 2D Systems with SBIC). *Consider the autonomous nonlinear 2D system* (1)*. The origin is globally asymptotically stable, if for any L'_p and L''_∞ Smooth Bounded Initial Conditions (according to Definition 4) it is stable, and the limit $\lim_{t+k\to\infty} x(t,k) = 0$ holds for all solutions $x(t,k)$.*

Definition 8 (Exponential Stability of Nonlinear 2D Systems). *Consider the autono-*

mous nonlinear 2D system (1). *The origin is globally exponentially stable, if for any exponentially decaying initial conditions there exist positive constants η_{c} and $\eta_{\mathrm{d}} < 1$, and $M_{\mathrm{c}}, M_{\mathrm{d}} < \infty$ such that the conditions $|x_{\mathrm{c}}(t,k)| \leq M_{\mathrm{c}} e^{-\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k}$ and $|x_{\mathrm{d}}(t,k)| \leq M_{\mathrm{d}} e^{-\eta_{\mathrm{c}} t} \eta_{\mathrm{d}}^{k}$ hold for all solutions $x(t,k)$.*

AppendixB. Lemmas

The following Lemma is needed to show Lyapunov stability of nonlinear 2D continuous-discrete systems.

Lemma 4. *[11] Consider the 2D space of two variables t and k and the 2D vector field $V^{\mathrm{T}}(t,k) = (V_{\mathrm{c}}(t,k), V_{\mathrm{d}}(t,k))$ with $V_{\mathrm{c}} \geq 0$ and $V_{\mathrm{d}} \geq 0$ for all t and k. If the divergence of the vector field satisfies* $\operatorname{div} V \leq 0$ *for all t and k, then*

$$\sum_{l=0}^{k} V_{\mathrm{c}}(t,l) \leq \sum_{l=0}^{k} V_{\mathrm{c}}(0,l) + \int_{0}^{t} V_{\mathrm{d}}(\tau,0)\mathrm{d}\tau \quad \textit{and} \tag{B.1}$$

$$\int_{0}^{t} V_{\mathrm{d}}(\tau,k)\mathrm{d}\tau \leq \sum_{l=0}^{k} V_{\mathrm{c}}(0,l) + \int_{0}^{t} V_{\mathrm{d}}(\tau,0)\mathrm{d}\tau \quad \forall t,k > 0. \tag{B.2}$$

The following corollary shows that if a suitable regular 2D Lyapunov function exists, the integral $\int_0^\infty V_{\mathrm{c}}(x_{\mathrm{c}}(t,k))\mathrm{d}t$ and the sum $\sum_{k=0}^{\infty} V_{\mathrm{d}}(x_{\mathrm{d}}(t,k))$ are bounded. This result together with Lemmas 6 and 7 facilitates the proof of asymptotic stability in Section 3.

Corollary 5. *Consider the nonlinear 2D system in* (1). *If there exists a regular 2D Lyapunov function V according to Definition 2 and the initial conditions are L_V and L_∞ bounded according to Definition 3, then there exist $\overline{M}_{\mathrm{c}}, \overline{M}_{\mathrm{d}} < \infty$ independent of k and t, such that*

$$\int_{0}^{\infty} V_{\mathrm{c}}(x_{\mathrm{c}}(t,k))\mathrm{d}t \leq \overline{M}_{\mathrm{c}}, \quad \textit{and} \quad \sum_{k=0}^{\infty} V_{\mathrm{d}}(x_{\mathrm{d}}(t,k)) \leq \overline{M}_{\mathrm{d}}. \tag{B.3}$$

Proof. From the definition of the regular 2D Lyapunov function

$$V_{\mathrm{c}}(x_{\mathrm{c}}(t,k)) \leq \mathrm{e}^{-a_{\mathrm{c}}t} V_{\mathrm{c}}(x_{\mathrm{c}}(0,k)) + b_{\mathrm{c}} \int_{0}^{t} \mathrm{e}^{-a_{\mathrm{c}}\tau} V_{\mathrm{d}}(x_{\mathrm{d}}(t-\tau,k))\mathrm{d}\tau \tag{B.4}$$

and thus

$$\begin{aligned}\int_{0}^{\infty} V_{\mathrm{c}}(x_{\mathrm{c}}(t,k))\mathrm{d}t \leq & V_{\mathrm{c}}(x_{\mathrm{c}}(0,k)) \int_{0}^{\infty} \mathrm{e}^{-a_{\mathrm{c}}t}\mathrm{d}t \\ & + b_{\mathrm{c}} \int_{0}^{\infty} \int_{0}^{t} \mathrm{e}^{-a_{\mathrm{c}}\tau} V_{\mathrm{d}}(x_{\mathrm{d}}(t-\tau,k))\mathrm{d}\tau\mathrm{d}t.\end{aligned} \tag{B.5}$$

Since the initial conditions are L_∞ bounded and $a_{\mathrm{c}} > 0$, the first term of the right hand side of (B.5) can be bounded from above by $\frac{V_{\mathrm{c}}(\zeta_{\mathrm{c}})}{a_{\mathrm{c}}}$. Using the fact that the convolution is commutative and interchanging the order of integration in the second term of the right hand side of (B.5) yields

$$
\begin{aligned}
& b_{\mathrm{c}} \int_0^\infty \int_\tau^\infty \mathrm{e}^{-a_{\mathrm{c}}(t-\tau)} V_{\mathrm{d}}(x_{\mathrm{d}}(\tau,k))\mathrm{d}t\mathrm{d}\tau \\
\leq & b_{\mathrm{c}} \int_0^\infty V_{\mathrm{d}}(x_{\mathrm{d}}(\tau,k)) \cdot \left(\int_\tau^\infty \mathrm{e}^{-a_{\mathrm{c}}(t-\tau)} \mathrm{d}t \right) \mathrm{d}\tau \leq \frac{b_{\mathrm{c}}}{a_{\mathrm{c}}} \int_0^\infty V_{\mathrm{d}}(x_{\mathrm{d}}(t,k))\mathrm{d}t. \qquad \text{(B.6)}
\end{aligned}
$$

Since the divergence is nonpositive we can apply Lemma 4. As the initial conditions are in L_V, the bound $\overline{M}_{\mathrm{c}}$ is $\overline{M}_{\mathrm{c}} := \frac{V_{\mathrm{c}}(\zeta_{\mathrm{c}})}{a_{\mathrm{c}}} + \frac{b_{\mathrm{c}}}{a_{\mathrm{c}}}(\xi_{\mathrm{c}} + \xi_{\mathrm{d}})$. The existence of $\overline{M}_{\mathrm{d}}$ can be shown in the similar way.

It is further shown below that the first derivatives of the states x_{c} and x_{d} with respect to t and the differences with respect to k are bounded if the state space equations, the initial conditions and the Lyapunov function fulfil certain differentiability criteria.

Lemma 6. *Consider the nonlinear 2D system* (1) *and the related time-varying linear system with*

$$
\begin{pmatrix} \dot{y}_{\mathrm{c}}(t,k) \\ \Delta y_{\mathrm{d}}(t,k) \end{pmatrix} = \begin{bmatrix} F_{\mathrm{cc}}(y_{\mathrm{c}},y_{\mathrm{d}}) & F_{\mathrm{cd}}(y_{\mathrm{c}},y_{\mathrm{d}}) \\ F_{\mathrm{dc}}(y_{\mathrm{c}},y_{\mathrm{d}}) & F_{\mathrm{dd}}(y_{\mathrm{c}},y_{\mathrm{d}}) \end{bmatrix} \begin{pmatrix} y_{\mathrm{c}}(t,k) \\ y_{\mathrm{d}}(t,k) \end{pmatrix} \qquad \text{(B.7)}
$$

with the Jacobian matrices $F_{il} := \left.\frac{\partial f_i(x_{\mathrm{c}},x_{\mathrm{d}})}{\partial x_l}\right|_{x_{\mathrm{c}}=x_{\mathrm{c}}(t,k),x_{\mathrm{d}}=x_{\mathrm{d}}(t,k)}$ *for* $i,l \in \{c,d\}$. *If there exists a regular 2D Lyapunov function V according to Definition 2 for system* (1) *and in addition there exist two Lyapunov functions W_{c} and W_{d} and scalars* $1 \leq p < \infty$ *and* $0 < \alpha'_{\mathrm{c}},\alpha'_{\mathrm{d}},\underline{\alpha}'_{\mathrm{c}},\underline{\alpha}'_{\mathrm{d}},\overline{\alpha}'_{\mathrm{c}},\overline{\alpha}'_{\mathrm{d}}, a_{\mathrm{c}},a_{\mathrm{d}},b_{\mathrm{c}},b_{\mathrm{d}} < \infty$ *such that the initial conditions are L'_p and L''_∞ smooth bounded according to Definition 4 and*

$$
\begin{aligned}
\underline{\alpha}'_{\mathrm{c}}|y_{\mathrm{c}}|^p \leq & W_{\mathrm{c}}(y_{\mathrm{c}}) \leq \overline{\alpha}'_{\mathrm{c}}|y_{\mathrm{c}}|^p, \quad \underline{\alpha}'_{\mathrm{d}}|y_{\mathrm{d}}|^p \leq W_{\mathrm{d}}(y_{\mathrm{d}}) \leq \overline{\alpha}'_{\mathrm{d}}|y_{\mathrm{d}}|^p, && \text{(B.8)} \\
\dot{W}_{\mathrm{c}}(y_{\mathrm{c}}) \leq & - a_{\mathrm{c}} W_{\mathrm{c}}(y_{\mathrm{c}}) + b_{\mathrm{c}} W_{\mathrm{d}}(y_{\mathrm{d}}), && \text{(B.9)} \\
\dot{W}_{\mathrm{d}}(y_{\mathrm{d}}) \leq & - a_{\mathrm{d}} W_{\mathrm{d}}(y_{\mathrm{d}}) + b_{\mathrm{d}} W_{\mathrm{c}}(y_{\mathrm{c}}) && \text{(B.10)} \\
\mathrm{div} V(t,k) \leq & - \alpha'_{\mathrm{c}} \, |\dot{x}_{\mathrm{c}}(t,k)|_p^p - \alpha'_{\mathrm{d}} \, |\Delta x_{\mathrm{d}}(t,k)|_p^p && \text{(B.11)}
\end{aligned}
$$

for all $t,k > 0$*, then*

1. *the first derivative and difference of $x_{\mathrm{c}}(t,k)$ and $x_{\mathrm{d}}(t,k)$ are in* L_∞ $[0,\infty) \times [0,\infty)$ *and* L_p $[0,\infty) \times [0,\infty)$*, i.e. there exist* $M_{i\mathrm{c}},M_{i\mathrm{d}},\overline{M}_{i\mathrm{c}},\overline{M}_{i\mathrm{d}} < \infty$ *such that for* $i \in \{c,d\}$

$$
\sup_{(t,k)\in\mathbb{R}_{\geq 0}\times\mathbb{N}} |\dot{x}_i(t,k)| \leq M_{i\mathrm{c}}, \qquad \sup_{(t,k)\in\mathbb{R}_{\geq 0}\times\mathbb{N}} |\Delta x_i(t,k)| \leq M_{i\mathrm{d}}, \qquad \text{(B.12)}
$$

$$
\sum_{k=0}^\infty \int_0^\infty |\dot{x}_i(t,k)|^p \mathrm{d}t \leq \overline{M}_{i\mathrm{c}}, \quad \sum_{k=0}^\infty \int_0^\infty |\Delta x_i(t,k)|^p \mathrm{d}t \leq \overline{M}_{i\mathrm{d}}, \qquad \text{(B.13)}
$$

2. *the second derivatives and differences of $x_{\mathrm{c}}(t,k)$ and $x_{\mathrm{d}}(t,k)$ are in* L_∞ $[0,\infty) \times$

$[0,\infty)$*, i.e. there exist* $M_{icc}, M_{icd}, M_{idd} < \infty$ *such that for* $i \in \{c,d\}$

$$\sup_{(t,k)\in\mathbb{R}_{\geq 0}\times\mathbb{N}} |\ddot{x}_i(t,k)| \leq M_{icc}, \qquad \sup_{(t,k)\in\mathbb{R}_{\geq 0}\times\mathbb{N}} |\Delta^2 x_i(t,k)| \leq M_{idd}, \tag{B.14}$$

$$\sup_{(t,k)\in\mathbb{R}_{\geq 0}\times\mathbb{N}} |\Delta\dot{x}_i(t,k)| = \sup_{(t,k)\in\mathbb{R}_{\geq 0}\times\mathbb{N}} \left|\frac{\mathrm{d}}{\mathrm{d}t}\Delta x_i(t,k)\right| \leq M_{icd}. \tag{B.15}$$

Proof. (a): The origin is stable by Corollary 1 and therefore the states are bounded. The bounds M_{cc} and M_{dd} are equal to the bounds on f_{c} and f_{d} for all $|x_{\mathrm{c}}| \leq M_{\mathrm{c}}$ and $|x_{\mathrm{d}}| \leq M_{\mathrm{d}}$.

Combining B.11, the fundamental theorem of calculus and the fact that the initial conditions are in L_V yields

$$\alpha'_{\mathrm{c}} \|\dot{x}_{\mathrm{c}}(\cdot,\cdot)\|_p^p + \alpha'_{\mathrm{d}} \|\Delta x_{\mathrm{d}}(\cdot,\cdot)\|_p^p \leq -\sum_{k=0}^{\infty}\int_0^{\infty} \mathrm{div} V(t,k)\mathrm{d}t \leq \xi_{\mathrm{c}} + \xi_{\mathrm{d}}.$$

Thus, $\dot{x}_{\mathrm{c}}(t,k)$ and $\Delta x_{\mathrm{d}}(t,k)$ are also in $L_p\,[0,\infty)\times[0,\infty)$.

To show that the first difference Δx_{c} is also in L_p and L_∞, use

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta x_{\mathrm{c}}(t,k) = F_{\mathrm{cc}}\Delta x_{\mathrm{c}} + F_{\mathrm{cd}}\Delta x_{\mathrm{d}} = F_{\mathrm{cc}}\Delta x_{\mathrm{c}} + F_{\mathrm{cd}} f_{\mathrm{d}}. \tag{B.16}$$

Therefore, assumptions (B.8) and (B.9) lead to

$$\alpha'_{\mathrm{c}}|\Delta x_{\mathrm{c}}|^p \leq \mathrm{e}^{-a_{\mathrm{c}}t} W_{\mathrm{c}}(\Delta x_{\mathrm{c}}(0,k)) + b_{\mathrm{c}}\int_0^t \mathrm{e}^{-a_{\mathrm{c}}(t-\tau)} W_{\mathrm{d}}(\Delta x_{\mathrm{d}}(\tau,k))\mathrm{d}\tau. \tag{B.17}$$

Hence,

$$\begin{aligned}\sum_{k=0}^{\infty}\int_0^{\infty} \underline{\alpha}'_{\mathrm{c}}|\Delta x_{\mathrm{c}}|^p \mathrm{d}t \leq & \sum_{k=0}^{\infty}\int_0^{\infty} \mathrm{e}^{-a_{\mathrm{c}}t} W_{\mathrm{c}}(\Delta x_{\mathrm{c}}(0,k))\mathrm{d}t \\ & + b_{\mathrm{c}}\sum_{k=0}^{\infty}\int_0^{\infty}\int_0^t \mathrm{e}^{-a_{\mathrm{c}}(t-\tau)} W_{\mathrm{d}}(\Delta x_{\mathrm{d}}(\tau,k))\mathrm{d}\tau\mathrm{d}t.\end{aligned} \tag{B.18}$$

Using the fact that the initial conditions are L'_p the first term on the right hand side of (B.18) is bounded by $\frac{\overline{\alpha}'_{\mathrm{d}}\xi'_{\mathrm{c}}}{a_{\mathrm{c}}}$. To calculate an upper bound for the second term on the right hand side of (B.18) follow similar steps as in (B.6) to obtain the upper bound $\frac{b_{\mathrm{c}}\overline{\alpha}'_{\mathrm{d}}}{a_{\mathrm{c}}}\frac{\xi_{\mathrm{c}}+\xi_{\mathrm{d}}}{\alpha'_{\mathrm{d}}}$. Thus, the first difference Δx_{c} is in L_p. The same arguments can be followed to show that $\dot{x}_{\mathrm{d}}$ is in L_p. Note that (B.17) — together with the fact that the initial conditions' first derivatives and differences are in L_∞ and f_{c} and f_{d} are bounded — also proves that Δx_{c} and $\dot{x}_{\mathrm{d}}$ are in L_∞.

(b): To show that M_{ccc} exists observe that $\ddot{x}_{\mathrm{c}}(t,k) = F_{\mathrm{cc}}(x_{\mathrm{c}},x_{\mathrm{d}})f_c(x_{\mathrm{c}},x_{\mathrm{d}}) + F_{cd}(x_{\mathrm{c}},x_{\mathrm{d}})\dot{x}_{\mathrm{d}}(t,k)$. As all terms on the right hand side are bounded, M_{ccc} exists. The bound M_{ccd} exists since all terms on the right hand side of (B.16) are bounded. Similar arguments show that the remaining second derivatives and differences are in L_∞.

Lemma 6 proves that the system trajectories are sufficiently smooth given sufficiently smooth initial conditions. It will be shown next that a 2D version of Barbalat's Lemma, [16], applies to such systems with bounded trajectories and derivatives and differences. Both results are needed to prove asymptotic stability of 2D continuous-discrete systems in Section 3.

Lemma 7. *[10, Lemma 5] Consider the 2D function* $f : \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$. *If* $f(t,k)$ *is both in* $L_p\ [0,\infty)\times[0,\infty)$ *and* $L_\infty\ [0,\infty)\times[0,\infty)$ *and its derivative* $\dot{f}(t,k)$ *and its difference* $\Delta f(t,k)$ *are in* $L_\infty\ [0,\infty)\times[0,\infty)$, *then* $\lim_{t,k\to\infty} f(t,k) = 0$ *and* $f(t,k)$ *is uniformly convergent in both directions, i.e. for all* $\epsilon > 0$ *there exists a* $T(\epsilon) < \infty$ *such that*

$$\forall (t,k) \in \{\mathbb{R}\times[T(\epsilon),\infty)\} \cup \{[T(\epsilon),\infty)\times\mathbb{Z}\} : |f(t,k)| < \epsilon.$$

References

[1] Angeli, D. [1999], 'Intrinsic robustness of global asymptotic stability', *Systems & Control Letters* **38**(4-5), 297–307.

[2] Angeli, D., Sontag, E. and Wang, Y. [2000], 'A Characterization of Integral Input-to-State Stability', *IEEE Trans. on Autom. Control* **45**.

[3] Bernstein, D. S. [2005], *Matrix Mathematics*, Princeton University Press.

[4] Bouagada, D. and Van Dooren, P. [2011], 'On the stability of 2D state-space models', *Numerical Linear Algebra with Applications* .

[5] El-Agizi, N. and Fahmy, M. [1979], 'Sufficient conditions for the nonexistence of limit cycles in two-dimensional digital filters', *IEEE Transactions on Circuits and Systems* **26**(6), 402–406.

[6] Hinamoto, T. [1993], '2-D Lyapunov Equation and Filter Design Based on the Fornasini-Marchesini Second Model', *IEEE Transactions on Circuits and Systems* **40**(2), 102–110.

[7] Kar, H. and Singh, V. [2001], 'Stability analysis of 2-D digital filters described by the fornasini–marchesini second model using overflow nonlinearities', *IEEE Transactions on Circuits and Systems - I: Fundamental Theory andApplications* **48**(5), 612–617.

[8] Kar, H. and Singh, V. [2005], 'Stability analysis of 2-d digital filters with saturation arithmetic: An LMI approach', *IEEE Trans. on Signal Proc.* **53**(6).

[9] Knorn, S. [2013], A 2D Systems Stability Analysis of Vehicle Platoons, PhD thesis, National University of Ireland Maynooth.

[10] Knorn, S. and Middleton, R. H. [2013*a*], 'Stability of Two-Dimensional Linear Systems with Singularities on the Stability Boundary using LMIs', *IEEE Transactions on Automatic Control* **58**(10).

[11] Knorn, S. and Middleton, R. H. [2013*b*], Two-dimensional analysis of string stability of nonlinear vehicle strings, *in* '52nd IEEE Conference on Decision and Control', pp. 5864–5869.

[12] Kurek, J. E. [1995], 'Stability of nonlinear parameter-varying digital 2-D systems', *IEEE Transactions on Automatic Control* **40**(8).

[13] Kurek, J. E. [2012], 'Stability of nonlinear time-varying digital 2-D Fornasini-Marchesini system', *Multidim. Systems and Signal Processing* .

[14] Liu, D. [1998], 'Lyapunov Stability of Two-Dimensional Digital Filters with Overflow Nonlinearities', *IEEE Transactions on Circuits and Systems - I: Fundamental Theory andApplications* **45**(5).

[15] Liu, D. and Michel, A. N. [1994], 'Stability Analysis of State-Space Realizations for Two-Dimensional Filters with Overflow Nonlinearities', *IEEE Transactions on Circuits and Systems - I: Fundamental Theory andApplications* **41**(2), 127–137.

[16] Logemann, H. and Ryan, E. P. [2004], 'Asymptotic Behaviour of Nonlinear Systems', *The American Mathematical Monthly* **111**(10).

[17] Pandolfi, L. [1984], 'Exponential Stability of 2-D Systems', *Systems & Control Letters* **4**(6), 381–385.

[18] Singh, V. [2005], 'Robust Stability of 2-D Digital Filters Employing Saturation', *IEEE Signal Processing Letters* **12**(2), 142–145.

[19] Singh, V. [2007], 'Improved Criterion for Global Asymptotic Stability of 2-D Discrete Systems With State Saturation', *IEEE Signal Processing Letters* **14**(10), 719–722.

[20] Singh, V. [2008], 'Stability Analysis of 2-D Discrete Systems Described by the Fornasini–Marchesini Second Model With State Saturation', *IEEE Trans. on Circuits and Systems - II: Express Briefs* **55**(8).

[21] Tzafestas, S., Kanellakis, A. and Theodorou, N. [1992], 'Two-dimensional digital filters without overflow oscillations and instability due to finite word length', *IEEE Transactions on Signal Processing* **40**(5).

[22] Vidyasagar, M. [2002], *Nonlinear Systems Analysis*, Prentice-Hall.

[23] Xiao, C. and Hill, D. J. [1996], 'Stability and Absence of Overflow Oscillations for 2-D Discrete-Time Systems', *IEEE Transactions on Signal Processing* **44**(8), 2108–2110.

[24] Yanakiev, D. and Kanellakopoulos, I. [1995], Variable Time Headway for String Stability of Automated Heavy-Duty Vehicles, *in* '34th Conference on Decision & Control', pp. 4077–4081.

[25] Zhu, Q. and Hu, G.-D. [2011], 'Stability and absolute stability of a general 2-D non-linear FM second model', *IET Control Theory & Applications* **5**(1), 239–246.

[26] Zhu, Q., Hu, G.-D. and Yin, Y.-X. [2012], 'Lyapunov-Type Theorem of General Two-Dimensional Nonlinear Parameter-Varying FM Second Model', *IEEE Trans. on Circuits and Systems - II: Express Briefs* **59**.