Quadratic stability and singular SISO switching systems

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Abstract

In this note we consider the problem of determining necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for a pair of stable linear time-invariant systems whose system matrices are of the form A, $A - gh^T$, and where one of the matrices is singular. A necessary and sufficient condition for the existence of such a function is given in terms of the spectrum of the product $A(A - gh^T)$. Examples are presented to illustrate our result.

1 Introduction

Consider a switching system described by

$$\dot{x} = [A - \sigma(t,x)gh^T]x \tag{1}$$

where the state $x(t)$ and g, h are real vectors, A is a real square matrix, and the scalar switching function σ satisfies

$$0 \leq \sigma(t,x) \leq 1\,. \tag{2}$$

Suppose A is a Hurwitz matrix, that is, all its eigenvalues have negative real parts; then the system corresponding to $\sigma(t,x) \equiv 0$, that is, $\dot{x} = Ax$, is globally asymptotically stable about the origin of the state space. Suppose also that all the eigenvalues of $A - gh^T$ have negative real parts except for a single eigenvalue at zero. Then the system corresponding to $\sigma(t,x) \equiv 1$, that is, $\dot{x} = (A - gh^T)x$, is stable (but not asymptotically stable) about the origin and all its solutions are bounded. We can guarantee that the switching system (1) is stable about the origin and all solutions are bounded if there is a real symmetric positive definite matrix P satisfying the following two matrix inequalities.

$$A^TP + PA \;<\; 0 \tag{3}$$

$$(A - gh^T)^TP + P(A - gh^T) \;\leq\; 0\,. \tag{4}$$

Also, the stability/boundedness properties are guaranteed for any switching function provided it satisfies the constraint (2). Satisfaction of the above inequalities implies that the quadratic function $V(x) = x^T Px$ is a Lyapunov function for the system in the sense that along all solutions, one has $\dot{V} \leq 0$; this guarantees the desired stability/boundedness properties. We refer to a matrix $P = P^T > 0$ satisfying (3) as a commmon Lyapunov matrix for A and $A - gh^T$; the corresponding Lyapunov function is referred to as a common Lyapunov function.

Such stability problems arise in a variety of applications; for example, in applications where integrators are switched in and out of feedback loops to achieve certain performance objectives [**?**, **?**].

In this short note, we show that the following simple condition is both necessary and sufficient condition for the existence of a common Lyapunov matrix P.

The matrix product $A(A - gh^T)$ has no eigenvalues with negative real part and only one zero eigenvalue.

In the next section we present some results on positive real transfer functions which are useful in the development of the main result. In particular, Theorem 2.1 provides a simple spectral characterization of strictly positive real transfer functions. We believe this is a useful result on its own and not just for the purposes of obtaining the main result. Section **??** develops the main result of this paper. To achieve this we also need the Kalman-Yacubovich-Popov lemma for proper SISO systems that is found in most textbooks; see the book by Boyd [1] or Khalil [2]. Throughout, known results are quoted without proof whereas new results are given with full proofs.

2 Strictly positive real transfer functions

Before obtaining our main result, we obtain some preliminary results on strictly positive real (SPR) transfer functions. In everything that follows, A is a real $n \times n$ matrix and b, c are vectors in real n-vectors.

Recall that a scalar transfer function H is strictly positive real (SPR) if there exists a scalar $\alpha > 0$ such that H is analytic in the region of the complex plane for which $Re(s) \geq 0$ and

$$H(\jmath\omega - \alpha) + H(\jmath\omega - \alpha)^* \geq 0 \qquad \text{for all} \qquad \omega \in \mathbb{R}\,. \tag{5}$$

We say H is SPR if $H(\jmath\omega) + H(\jmath\omega)^*$ is not identically zero for all $\omega \in \mathbb{R}$. For convenience, we will include regularity as a requirement for SPR.

Lemma 2.1 *[2] Suppose A is Hurwitz. Then the transfer function $H(s) = c^T(sI - A)^{-1}b$ is SPR if and only if*

$$H(j\omega) + H(j\omega)^* \;>\; 0 \qquad \textit{for all } \omega \in \mathbb{R} \tag{6}$$

$$\lim_{\omega \to \infty} \omega^2 \big(H(j\omega) + H(j\omega)^*\big) \;>\; 0. \tag{7}$$

Lemma 2.2 *The transfer function $H(s) = c^T(sI - A)^{-1}b$ is SPR if and only if the transfer function $H_I(s) = c^T(sI - A^{-1})^{-1}b$ is SPR.*

Proof : Suppose H is SPR. The identity $(sI - A^{-1})^{-1} = s^{-1}I - s^{-2}(s^{-1}I - A)^{-1}$ implies that

$$H_I(s) = s^{-1}c^T b - s^{-2}H(s^{-1})\,; \tag{8}$$

hence, whenever $\omega \neq 0$,

$$H_I(\jmath\omega) + H_I(\jmath\omega)^* = \omega^{-2}(H(-\jmath\omega^{-1}) + H(-\jmath\omega^{-1})^*) > 0$$

Considering limits as $\omega \to 0$,

$$H_I(0) + H_I(0)^* = \lim_{\tilde{\omega}\to\infty} \tilde{\omega}^2(H(\jmath\tilde{\omega}) + H(\jmath\tilde{\omega})^*) > 0$$

Finally, we note that

$$\lim_{\omega\to\infty} \omega^2(H_I(\jmath\omega) + H_I(\jmath\omega)^*) = H(0) + H(0)^* > 0\,.$$

□

The core of our main result is based on a spectral condition for strict positive realness [3]. This result follows as an immediate consequence of the following lemma.

Lemma 2.3 *[4, 5, 6] Let $H(s) = d + c^T(sI - A)^{-1}b$ where A is invertible. Then, $H(s^{-1}) = \bar{d} + \bar{c}^T(sI - \bar{A})^{-1}\bar{b}$ with $\bar{A} = A^{-1}$, $\bar{b} = -A^{-1}b$, $\bar{c}^T = c^TA^{-1}$ and $\bar{d} = d - c^TA^{-1}b$.*

Proof : Using the definitions in the lemma,

$$\begin{aligned}
\bar{d} + \bar{c}^T(sI - \bar{A})^{-1}\bar{b} &= d - c^TA^{-1}b - c^TA^{-1}(sI - A^{-1})^{-1}A^{-1}b \\
&= d - c^TA^{-1}\left(I + (sI - A^{-1})^{-1}A^{-1}\right)b \\
&= d - c^TA^{-1}(sA - I)^{-1}sAb \\
&= d + c^T(s^{-1}I - A)^{-1}b,
\end{aligned}$$

which proves the assertion of the lemma.

Comment : Note that when H is SPR we must have $\bar{d} > 0$. This follows from the fact that $\bar{d} = H(0)$ and $H(0) + H(0)^* > 0$ since H is SPR.

Now we give the aforementioned spectral characterisation of strict positive realness.

Theorem 2.1 *Suppose A is Hurwitz. Then, the following statements are equivalent.*

(a) The transfer function $H(s) = c^T(sI - A)^{-1}b$ is SPR.

(b) $c^TA^{-1}b < 0$ *and the matrix product* $A^{-1}(A^{-1} - \frac{A^{-1}bc^TA^{-1}}{c^TA^{-1}b})$ *has no negative real eigenvalues and exactly one zero eigenvalue.*

(c) $c^TAb < 0$ *and the matrix product* $A(A - \frac{Abc^TA}{c^TAb})$ *has no negative real eigenvalues and exactly one zero eigenvalue.*

Proof : In what follows it is convenient to work with $H(s^{-1})$ as in Lemma 2.3. In particular, the conditions for SPR of H may be restated in terms of the transfer function $H(s^{-1})$. Specifically, conditions (6) and (7) for SPR are equivalent to

$$\lim_{\omega\to\infty} H(-j\omega^{-1}) + H(-j\omega^{-1})^* \;>\; 0 \tag{9}$$

$$H(-j\omega^{-1}) + H(-j\omega^{-1})^* \;>\; 0 \qquad \forall\, \omega \in \mathbb{R} \quad \omega \neq 0 \tag{10}$$

$$\lim_{\omega\to 0} \frac{1}{\omega^2}\left[H(-j\omega^{-1}) + H(-j\omega^{-1})^*\right] \;>\; 0 \tag{11}$$

Condition (9) is equivalent to $c^TA^{-1}b < 0$.

Now consider conditions (10) and (11). Since $\bar{A}$ is invertible, Lemma 2.3 tells us that

$$H(-j\omega^{-1}) = \bar{d} + \bar{c}^T(j\omega I - \bar{A})^{-1}\bar{b} \tag{12}$$

with $\bar{A}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ defined in Lemma 2.3. Using the results in [7] we have

$$\bar{c}^T(j\omega I - \bar{A})^{-1}\bar{b} + \left[\bar{c}^T(j\omega I - \bar{A})^{-1}\bar{b}\right]^* = -2\bar{c}^T(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{b}$$

Since $\bar{d} = -c^TA^{-1}b > 0$, we can write

$$\begin{aligned} H(-j\omega^{-1}) + H(-j\omega^{-1})^* &= 2\bar{d}\det\left[1 - \frac{1}{\bar{d}}\bar{c}^T(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{b}\right] \\ &= 2\bar{d}\det\left[I - \frac{1}{\bar{d}}(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{b}\bar{c}^T\right] \\ &= 2\bar{d}\det\left[(\omega^2 I + \bar{A}^2)^{-1}\right]\,\det\left[\omega^2 I + \bar{A}^2 - \frac{1}{\bar{d}}\bar{A}\bar{b}\bar{c}^T\right] \end{aligned}$$

Thus,

$$H(-j\omega^{-1}) + H(-j\omega^{-1})^* = \frac{2\bar{d}\det[\omega^2 I + M]}{\det\left[\omega^2 I + \bar{A}^2\right]}\,. \tag{13}$$

where

$$M := \bar{A}\left(\bar{A} - \frac{1}{\bar{d}}\bar{b}\bar{c}^T\right) = A^{-1}\left(A^{-1} - \frac{A^{-1}bc^TA^{-1}}{c^TA^{-1}b}\right).$$

Since A is Hurwitz, all the real eigenvalues of $\bar{A}^2 = A^{-2}$ are positive which implies that $\det[\omega^2 I + \bar{A}^2] \neq 0$ for all ω. Noting that $\det\left[\omega^2 I + \bar{A}^2\right] > 0$ for ω sufficiently large, it follows from continuity arguments that $\det[\omega^2 I + \bar{A}^2] > 0$ for all ω. Recalling that $\bar{d} > 0$ it follows from the above identity (13) that conditions (10) and (11) on $H(-\jmath\omega^{-1})$ are respectively equivalent to

$$\begin{aligned} \det[\omega^2 I + M] &> 0 \qquad \text{for all} \quad \omega \in \mathbb{R}, \quad \omega \neq 0 \\ \lim_{\omega \to 0} \frac{1}{\omega^2} \det[\omega^2 I + M] &> 0\,. \end{aligned}$$

Since, $\det[\omega^2 I + M] > 0$ for large ω, the above conditions are equivalent to

$$\det[\lambda I - M] \neq 0 \qquad \text{for all} \quad \lambda \in \mathbb{R}, \quad \lambda < 0 \tag{14}$$

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \det[\lambda I - M] \neq 0\,. \tag{15}$$

Condition (14) is equivalent to the requirement that M has no negative real eigenvalues. Since $Mb = 0$ and $b \neq 0$, the matrix M must have at least one zero eigenvalue; hence $\det[\lambda I - M] = \lambda q(\lambda)$ and all the other eigenvalues of M are given by the roots of the polynomial q. Thus condition (15) is equivalent to $q(0) \neq 0$, that is, zero is not a root of q. Thus (15) is equivalent to the requirement that M has only one eigenvalue at zero.

The equivalence between the first and third statement of the lemma follows from the SPR equivalence of $c^T(sI - A^{-1})^{-1}b$ and $c^T(sI - A)^{-1}b$ as stated in Lemma 2.2.

3 Main result

In everything that follows, A is a real $n \times n$ matrix and g and h are real n-vectors.

The proof of the main result requires the following KYP lemma.

Lemma 3.1 *[1] Suppose (A, b) is controllable and (A, c) is observable. Then, the following statements are equivalent.*

(i) The matrix A is Hurwitz and the transfer function $H(s) = c^T(sI - A)^{-1}b$ is SPR.

(ii) There exists a matrix $P = P^T > 0$ that satisfies the constrained Lyapunov inequality:

$$\begin{aligned} A^T P + PA &< 0 \\ Pb &= c\,. \end{aligned}$$

(iii) There exists a matrix $P = P^T > 0$ such that the following Lyapunov inequalities are satisfied:

$$\begin{aligned} A^T P + PA &< 0 \\ -\left(cb^T P + Pbc^T\right) &\leq 0. \end{aligned}$$

Comments : The best discussion of a strictly positive real transfer function can still be found in Narendra & Taylors book on Frequency domain stability criteria [8]. The assumption that (A, c) is observable ensures that P is positive definite in the theorem [9].

Theorem 3.1 (Main Theorem) *Suppose that A is Hurwitz and all the eigenvalues of $A - gh^T$ have negative real part, except one, which is zero. Suppose also that (A, g) is controllable and (A, c) is observable. Then, there exists a matrix $P = P^T > 0$ such that*

$$A^T P + PA < 0 \tag{16}$$

$$(A - gh^T)^T P + P(A - gh^T) \leq 0 \tag{17}$$

if and only if the matrix product $A(A - gh^T)$ has no real negative eigenvalues and exactly one zero eigenvalue.

Proof : The proof consists of two parts. First we use an equivalence to show that the conditions on $A(A - gh^T)$ are sufficient for the existence of a Lyapunov matrix P with the required properties. We then show that these conditions are also necessary.

Sufficiency : Let $c = A^{-T}h$ and let b be a right eigenvector of $A - gh^T$ corresponding to the zero eigenvalue. Then $b \neq 0$, $Ab = gh^T b = h^T bg$ and $c^T Ab = h^T b$. Since A is Hurwitz, we must have $h^T b \neq 0$, otherwise $Ab = 0$. Hence $c^T Ab \neq 0$ and, without loss of generality, we assume that b is chosen so that $c^T Ab = -1$. In this case,

$$g = -Ab \qquad \text{and} \qquad h^T = c^T A\,.$$

Controllability of (A, b) and observability of (A, c) follow from controllability of (A, g) and observability of (A, h), respectively.

Noting that

$$Z := A - gh^T = A - \frac{Abc^T A}{c^T Ab}\,,$$

it follows from Theorem 2.1 that the conditions on AZ imply that the transfer function $c^T(sI - A)^{-1}b$ is Strictly Positive Real. Consequently, it follows from Lemma 3.1 that there exists a matrix $P = P^T > 0$ such that

$$A^T P + PA < 0 \tag{18}$$

$$Pb = c\,. \tag{19}$$

Pre- and post- multiplying the above inequality by A^{-T} and A^{-1} shows that this inequality is equivalent to

$$A^{-T} P + PA^{-1} < 0 \tag{20}$$

This last inequality and (19) imply that

$$\begin{bmatrix} A^{-T}P + PA^{-1} & Pb - c \\ b^T P - c^T & 0 \end{bmatrix} \leq 0 \tag{21}$$

that is,

$$\begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix} \leq 0\,.$$

Since $c^T Ab = -1 \neq 0$,

$$\begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A - \frac{Abc^T A}{c^T Ab} & -\frac{Ab}{c^T Ab} \\ \frac{c^T A}{c^T Ab} & \frac{1}{c^T Ab} \end{bmatrix} = \begin{bmatrix} A - gh^T & -g \\ -h^T & -1 \end{bmatrix},$$

Post- and pre-multiplying inequality (21) by the above inverse and its transpose results in

$$\begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix}^{-T} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix}^{-1} \leq 0.$$

that is,

$$\begin{bmatrix} (A - gh^T)^T P + P(A - gh^T) & -Pg - h \\ -g^T P - h^T & -2 \end{bmatrix} \leq 0 \tag{22}$$

It immediately follows that for the above inequality to hold, we must have

$$(A - gh^T)^T P + P(A - gh^T) \leq 0. \tag{23}$$

Necessity : We first show that if there exists a matrix $P = P^T > 0$ satisfying conditions (16)-(17), then AZ cannot have a negative real eigenvalue. Note that the conditions on P are equivalent to

$$A^{-T}P + PA^{-1} < 0 \tag{24}$$

$$Z^T P + PZ \leq 0 \tag{25}$$

Hence, for any $\gamma > 0$,

$$(Z + \gamma A^{-1})^T P + P(Z + \gamma A^{-1}) < 0\,.$$

Since $P = P^T > 0$ this Lyapunov inequality implies that $Z + \gamma A^{-1}$ must be Hurwitz and hence, non-singular. Thus $AZ + \gamma I$ is nonsingular for all $\gamma > 0$. This means that AZ cannot have a negative real eigenvalue.

We now show that AZ cannot have a zero eigenvalue whose multiplicity is greater that one. To this end introduce the matrix

$$\tilde{A}(k) = Z + kgh^T\,.$$

Then $A = \tilde{A}(1)$ and inequalities (16)-(17) hold if and only if

$$\tilde{A}(k)^T P + P\tilde{A}(k) < 0 \qquad (26)$$

$$Z^T P + PZ \leq 0 \qquad (27)$$

hold for all k sufficiently close to one. As we have seen above, this implies that $A(k)Z$ cannot have negative real eigenvalues for all k sufficiently close to one. We shall show that AZ having an eigenvalue at the origin whose multiplicity is greater than one contradicts this statement.

By assumption, Z has a single eigenvalue at zero; a corresponding eigenvector is the vector b. Clearly, b is also an eigenvector corresponding to a zero eigenvalue of $A(k)Z$. Now choose any nonsingular matrix T whose first column is b. Then,

$$T^{-1}A(k)ZT = \begin{pmatrix} 0 & * \\ 0 & S + krs^T \end{pmatrix} \qquad (28)$$

and the eigenvalues of $A(k)Z$ consist of zero and the eigenvalues of $S + krs^T$. Note that the matrix S must be invertible since

$$T^{-1}Z^2T = T^{-1}A(0)ZT = \begin{pmatrix} 0 & * \\ 0 & S \end{pmatrix}$$

and Z^2 has only a single eigenvalue at zero. Now suppose that $AZ = A(1)Z$ has an eigenvalue at the origin whose multiplicity is greater than one. Then $S + rd^T$ must have a eigenvalue at zero; hence, $\det\left[S + rs^T\right] = 0$. Since S is invertible,

$$\det\left[S + krs^T\right] = \det[S]\ \det\left[I + kS^{-1}rs^T\right] = \det[S]\,(1 + ks^T S^{-1}r)\,,$$

and we must have $1 + s^T S^{-1}r = 0$ which implies that $s^T S^{-1}r = -1$. Hence,

$$\det\left[S + krs^T\right] = \det[S](1-k)\,.$$

Suppose $\det[S] > 0$. Then,

$$\det\left[S + krs^T\right] < 0$$

for $k > 1$. Since $\det\left[S + krs^T\right]$ is the product of all the eigenvalues of $S + krs^T$ and complex eigenvalues occur in complex conjugate pairs, $S + krs^T$ must have at least one real negative eigenvalue when $k > 1$. This yields the contradiction that $A(k)Z$ has a negative real eigenvalue when $k > 1$. The conclusion is the same for $\det[S] < 0$.

4 Examples

In this section we present two examples to illustrate the main features of our result.

Example 1 (No quadratic Lyapunov function) : Consider the dynamic systems $\Sigma_{A_1} : \dot{x} = A_1 x$ and $\Sigma_{A_2} : \dot{x} = A_2 x$ with:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad A_2 = A_1 - gh^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix},$$

with $g^T = [0, 0, 1]$ and $h^T = [-1, -1, -1]$. Note that A_1 is a Hurwitz matrix; whereas A_2 is singular with all its eigenvalues in the open left half of the complex plane, except one at the origin. Note also that (A, g) and (A, h) are controllable and observable.

The matrix product $A_1 A_2$ is given by:

$$A_1 A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \end{bmatrix}.$$

The eigenvalues of the matrix product $A_1 A_2$ are $(0, 0, 3)$. Hence, from the results of our main theorem, there cannot exist a $P = P^T > 0$ such that $A_1^T P + P A_1 < 0$ and $A_2^T P + P A_2 \leq 0$.

Example 2 (Quadratic stability) : Consider the dynamic systems $\Sigma_{A_1} : \dot{x} = A_1 x$ and $\Sigma_{A_2} : \dot{x} = A_2 x$ with:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.9 & -1.9 & -2.9 \end{bmatrix}, \quad A_2 = A_1 - gh^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix},$$

with $g^T = [0, 0, 1]$ and $h^T = [-0.9, -0.9, -0.9]$. Note that A_1 is a Hurwitz matrix; whereas A_2 is singular with all its eigenvalues in the open left half of the complex plane, except one at the origin. Note also that (A, g) and (A, h) are controllable and observable.

The matrix product $A_1 A_2$ is given by:

$$A_1 A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 2 & 3.9 \end{bmatrix}.$$

The eigenvalues of the matrix product $A_1 A_2$ are $(0, 0.0349, 2.8651)$. Hence, from the main theorem, there exist a $P = P^T > 0$ such that $A_1^T P + P A_1 < 0$ and $A_2^T P + P A_2 \leq 0$.

5 Concluding remarks

In this note we have derived necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for a pair of stable linear time-invariant systems whose system matrices are of the form A, $A - gh^T$, and where one of the matrices is singular. Future work will involve extending our results to non-quadratic stability criteria such as Popov.

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