Otto–von–Guericke–Universität Magdeburg

Diplomarbeit

Stability Issues in Distributed Systems of Vehicle Platoons

von

Steffi Klinge
(* 30. Jan. 1984 in Burg)

19. September 2008

Eingereicht an die: Otto–von–Guericke–Universität Magdeburg
Fakultät für Elektrotechnik und Informationstechnik
Universitätsplatz 2
39016 Magdeburg
Deutschland

Erstprüfer: Prof. Dietrich Flockerzi
Zweitprüfer: Prof. Achim Kienle

Max Planck Institut
Dynamik komplexer technischer Systeme
Sandtorstraße 1
39106 Magdeburg
Deutschland

Betreuer: Prof. Richard H. Middleton

Hamilton Institute
NUI Maynooth
Maynooth, Co. Kildare
Irland

Table of contents

List of figures

Outlines

The student will study and summarise background literature on string stability issues for platoons of vehicles. Following this, she will mathmetically model and simulate linear distributed feedback control systems illustrating string instability, and string stability for the case of a sufficiently large time headway. The student will also consider actuator saturation; and, design and simulate anti-windup controllers to compensate for actuator saturation. She will also examine stability of the resulting anti-windup schemes by using the circle criterion or Popov criterion if necessary. The student will also consider another nonlinear control scheme proposed in the literature, simulate such a scheme, and study whether stability proofs can be adapted for the nonlinear control scheme.

Preface

Acknowledgments

First I would like to express my deep gratitude to Prof. Richard H. Middleton for his outstanding and friendly support during the last semester, his endless patience answering all my questions and his helpful suggestions and commands.

For his thought-provoking impulses and helpful explanations on Lyapunov theory I am also indebted to Dr. Kai Wulff, who spent a lot of his time during his stay at the Hamilton Institute working on the string stability problem.

I would like to thank Prof. Robert Shorten very much for his encouraging enthusiasm and his numerous literature recommendations on avail.

For his constructive criticism, his motivation even in the most discouraging moments and, most of all, for his love I would like to thank Florian Knorn.

Furthermore, I would like to express my gratitude to Janine Holzmann, who helped me with all official matters far beyond her obligations.

I cannot list all the friendly people at the Hamilton Institute who create a welcoming and warm atmosphere despite the Irish weather. I greatly appreciate to work with them.

Thanks to my family and friends back home. Although I did not see them very often during the last six month they had time for me whenever I needed them.

Finally, I would like to thank the *Science Foundation Ireland* and the *Studienstiftung des Deutschen Volkes* as my stay in Ireland would not have been possible without their financial support.

Declaration of originality

I hereby declare that this thesis and the work reported herein was composed and originated entirely by myself. Information derived from the published and unpublished work of others is acknowledged in the text and a list of references is given in the bibliography.

Steffi Klinge

Introduction

With a significant increase in private and commercial road traffic, the past decades have witnessed a growing amount of research in the field of vehicle formation control. The higher traffic throughput by tighter "spacing", the safety gain or the reduced fuel consumption are but a few of the desirable features of platooning.

To that end, in 1966 Levine and Athans proposed an optimal controller to regulate a string of three vehicles, [17]. Later, Chu, [5], and Peppard,[24], investigated how disturbances can grow while propagating through the string and defined in that context the notion of "string stability". More Recently, a generalised definition of string stability was given in [31].

The analysis of string stability requires models of the constituent subsystems involved. While some authors focus on more realistic and detailed truck models, [10, 35], simpler linear vehicle models that are somewhat easier to handle can be found in [7, 20].

It has been shown that it is not possible to achieve string stability in a homogeneous string of strictly proper feedback control systems with nearest neighbour communications when using only linear systems with two integrators in the open loop and constant inter-vehicle spacing, [27]. In particular, the authors proposed a string stable solution where velocity and acceleration of the first vehicle in the string is transferred to all of the following vehicles. Seiler *et al.* showed that this result is independent of the particular controller design, [26].

Although string stability can be achieved with a speed dependent inter-vehicle spacing policy (also called 'time headway policy'), [4], the positive effects of platooning with regards to fuel economy or higher possible traffic throughput cannot be fully achieved since the steady state intervehicle spacing may become very large.

Other research was done on heterogeneous strings, i.e. the particular controller depends on the position within the string, [13, 16].

Yanakiev,[36], decreased inter-vehicle spacing and obtained string stability by introducing a variable time headway. An improvement in systems behaviour was also accomplished using a signed quadratic term in the controller, [36].

Note that although many of the applications of string stability are in the field of vehicle control, it may also be applied to different areas, such as interconnected water reservoirs in irrigation flow systems or supply chains.

In this thesis we will focus on a homogeneous string of vehicles. In the first chapter, we introduce two definition of string stability to analyse the behaviour of the string. We will also present a simple linear model to describe the vehicle dynamics and create a PID-controller for the subsystems.

As this linear approach with constant spacing will lead to string instability, we will, in Chapter 2, extend the controller to include a so-called “time headway” to solve this problem. We shall also investigate how to determine the minimal time headway that still guarantees string stability but minimises the inter-vehicle spacing. Simulation studies in this case confirm the string stability properties when sufficient time headway is included.

In Chapter 3 we will then analyse further approaches such as saturating the actuator signal in combination with an anti-windup scheme and a simple PIDQ-controller. We will show stability for the anti-windup scheme using the circle criterion. After displaying simulation results we will discus string stability for this approach.

The last chapter discusses a nonlinear expansion of the time headway introduced earlier. All controller approaches will be tested with two simulated manoeuvres and the results will be displayed in the corresponding parts of this work.

Chapter 1

Linear Controller

In this chapter we introduce a simple dynamic model of a vehicle. Also a common linear controller will be used to regulate the system. We will see, that this will lead us to string instability.

1.1 Introduction

Several authors (e.g. [5, 7, 20, 24, 26, 31]) discuss problems called *string instability*. Although they all refer to the same phenomenon where a small disturbance increases without bound while propagating through a homogeneous string of coupled linear dynamic systems different notations and definitions are used.

In this chapter two definitions based on [31] will be given. We will introduce a row of vehicles where each vehicle is modelled as a linear, two state model with time delay. We will further propose a common PID-controller to control each vehicle in the string.

Two manoeuvres in which every vehicle is required to keep a fixed distance to its predecessor and the first must follow a given trajectory will be discussed and simulated using the PID-controller introduced before.

We will see that even a small disturbance at the first vehicle grows while propagating down the string. To satisfy the requirements in the definitions of string stability to be introduced we will transform the system into deviation coordinates. For the transformed system we will show analytically why this phenomenon cannot be avoided and that we can never achieve string stability using a PID-controller and a fixed spacing policy between the vehicles.

1.2 Definition of String Stability

Loosely speaking, string instability is where a small disturbance at the beginning of the string grows without bound while propagating through the string.

A precise definition relating to this is given by Swaroop *et al.* in [31]. For a string of interconnected dynamic systems with bounded initial conditions they require bounded time signals of all states.

Here we want to study a special case of this definition where only the initial condition of the first subsystem in the string is non-zero and bounded. All other initial conditions need to be zero. (See Appendix A for more details on the norms used.)

Definition 1 (L_∞-String Stability)**.** *Consider a string of N dynamic systems of dimension n described by*

$$\dot{\xi}_i = f(\xi_i,\xi_{i-1}) \quad \forall 1 < i \leq N, \quad \dot{\xi}_1 = f(\xi_1,0) \tag{1.1}$$

where $N \in \mathbb{N}$, $\xi_i \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f(0,0) = 0$ and $\xi_i(0) = 0$, for $i = 2,\dots,N$. The origin $\xi_i = 0$, $\forall i$ of (1.1) is L_∞-string stable if given any $\epsilon > 0$ there exist a $\delta > 0$ that is independent of the string length N, such that

$$||\xi_1(0)||_\infty < \delta \Rightarrow \max_i ||\xi_i(t)||_{L_\infty} < \epsilon.$$

Definition 2 (L_2-String Stability)**.** *Consider a string of dynamic systems as described in (1.1). The origin $\xi_i = 0$, $\forall i$ of (1.1) is L_2-string stable if given any $\epsilon > 0$ there exist a $\delta > 0$ that is independent of the string length N, such that*

$$||\xi_1(0)||_2 < \delta \Rightarrow \sup_i ||\xi_i(t)||_{L_2} < \epsilon.$$

Note that for any fixed string length N both L_2- and L_∞-stability are equivalent. However, they may not be the same for an arbitrarily large N, as we shall see in the following.

Example (L_2- but not L_∞-string stable) : Consider the coupled system illustrated in Figure 1.1. We will see that under some conditions the system satisfies the definition of L_2-string stability but is not L_∞-string stable.

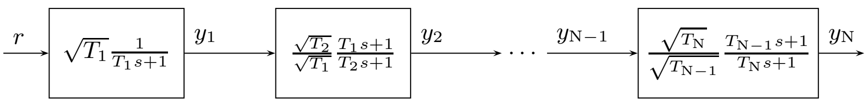


Figure 1.1: Example for string stability: Block diagram

Assume x_i is the state of the i^{th} subsystem and y_i its output. Let $T_1 = 1$ and $T_i > T_{i+1} > 0$ for all i such that T_N goes to zero as N goes to infinity. In the first subsystem, if the reference signal r is zero and the initial state of x_1 is $x_1(t = 0) = x_0$, then the state x_1 (and the output y_1) are

$$x_1(t) = y_1(t) = x_0 \mathrm{e}^{-t} \tag{1.2}$$

Assuming that all other initial conditions are zero, i.e. $x_i(t=0)=0$ for all $i \geq 2$, we can write y_N as

$$
\begin{aligned}
y_N(t) &= \mathcal{L}^{-1}\left\{ \frac{\sqrt{T_N}}{\cancel{\sqrt{T_{N-1}}}} \frac{\cancel{T_{N-1}s+1}}{T_N s+1} \frac{\cancel{\sqrt{T_{N-1}}}}{\sqrt{T_{N-2}}} \frac{T_{N-2}s+1}{\cancel{T_{N-1}s+1}} \cdots \right. \\
&\qquad\qquad\qquad\qquad \left. \cdots \frac{\sqrt{T_2}}{\sqrt{T_1}} \frac{T_1 s+1}{T_2 s+1} Y_1(s) \right\} \\
&= \mathcal{L}^{-1}\left\{ \sqrt{T_N} \frac{s+1}{T_N s+1} Y_1(s) \right\} \\
&= \frac{1}{\sqrt{T_N}} y_1(t) + \frac{1}{\sqrt{T_N}} \left(T_N - 1\right) x_N(t)
\end{aligned} \tag{1.3}
$$

In particular the N^{th} state evolves according to the following differential equation

$$
\dot{x}_N(t) = -\frac{1}{T_N} x_N(t) + \frac{1}{T_N} y_1(t) \tag{1.4}
$$

Integrating (1.4) yields

$$
\begin{aligned}
x_N(t) &= \mathrm{e}^{-\frac{t}{T_N}} \cancelto{0}{x_N(t=0)} + \int_0^t \frac{1}{T_N} \mathrm{e}^{-\frac{t-\tau}{T_N}} x_0 \mathrm{e}^{-\tau} \mathrm{d}\tau \\
&= \frac{1}{T_N} x_0 \mathrm{e}^{-\frac{t}{T_N}} \int_0^t \mathrm{e}^{\tau \frac{1-T_N}{T_N}} \mathrm{d}\tau \\
&= \frac{1}{T_N} x_0 \mathrm{e}^{-\frac{t}{T_N}} \frac{T_N}{1-T_N} \left[\mathrm{e}^{\tau \frac{1-T_N}{T_N}} \right]_0^t \\
&= \frac{1}{1-T_N} x_0 \mathrm{e}^{-t} - \frac{1}{1-T_N} x_0 \mathrm{e}^{-\frac{t}{T_N}}
\end{aligned} \tag{1.5}
$$

Substituting (1.2) and (1.5) in (1.3) results in

$$
\begin{aligned}
y_N(t) &= \frac{1}{\sqrt{T_N}} x_0 \mathrm{e}^{-t} + \frac{1}{\sqrt{T_N}} \left(T_N - 1\right) \frac{1}{1-T_N} x_0 \left(\mathrm{e}^{-t} - \mathrm{e}^{-\frac{t}{T_N}} \right) \\
&= \frac{1}{\sqrt{T_N}} x_0 \mathrm{e}^{-t} - \frac{1}{\sqrt{T_N}} x_0 \left(\mathrm{e}^{-t} - \mathrm{e}^{-\frac{t}{T_N}} \right) \\
&= \frac{1}{\sqrt{T_N}} x_0 \mathrm{e}^{-\frac{t}{T_N}}
\end{aligned} \tag{1.6}
$$

Since we assumed that T_N goes to zero as N grows the maximum of y_N will grow unbounded and Definition 1 would not hold. A simulation of this example with $T_i = 1/2^{i-1}$ and $x_0 = 1$ is shown in Figure 1.2.

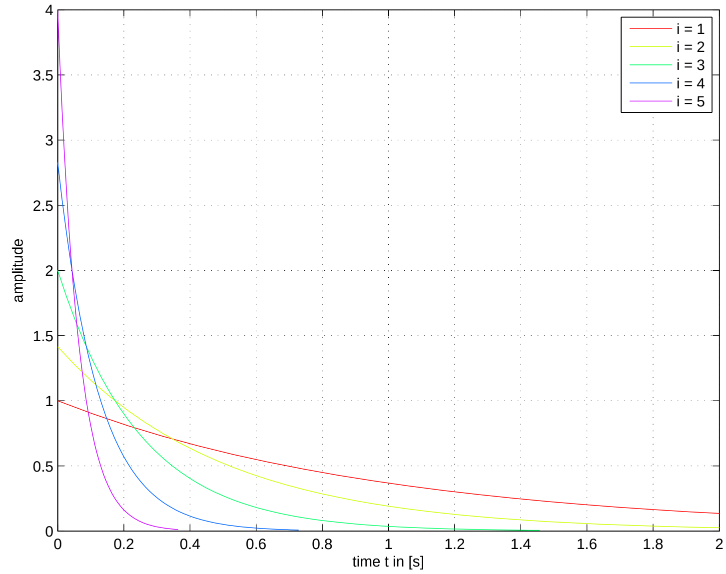


Figure 1.2: Example for string stability: Initial condition response

However, the system would satisfy Definition 2 since the L_2-Norm of y_N does not grow as N goes to infinity:

$$\begin{aligned} ||y_N(t)||_{L_2} &= \sqrt{\int_0^\infty y_N(t)^2 \mathrm{d}t} \\ &= \sqrt{\frac{1}{T_N} x_0^2 \int_0^\infty \mathrm{e}^{-\frac{2t}{T_N}} \mathrm{d}t} \\ &= \sqrt{\frac{1}{2} x_0^2 \left[-\mathrm{e}^{-\frac{2t}{T_N}} \right]_0^\infty} \\ &= \frac{x_0}{\sqrt{2}} \end{aligned} \tag{1.7}$$

Our restriction of Swaroop's definition is a reasonable simplification. As we will demonstrate later, all systems were transformed in order to obtain zero initial conditions for all but the first subsystem. Using Definition 1 and Definition 2 for string stability simplifies the analysis significantly and allows explicit analytic results.

Certainly, the definitions used here are weaker than the definition given by Swaroop. It may not always be possible to transform a system in order to meet the requirement of Definition 1 or Definition 2. It may be also possible to obtain string stability according to the definitions given here without satisfying Swaroop's definition. Hence the definitions of string stability provided here are necessary (but not sufficient) for string stability in the sense of Swaroop. However, we will in the following we will use the definitions given above since analytically we have not been able to prove string stability according to Swaroop.

1.3 Car Model

String instability is an open topic of research in several different domains and can be discussed for a number of applications. However, the theoretical discussion here is independent of any actual application. In this work we have chosen a string of vehicles to illustrate our results and test different controllers.

For the sake of simplicity, we consider a *homogeneous string* of vehicles, that is, one in which model and controller are identical for each subsystem. In this present work we will only use the following simplified second order model for each vehicle[1]

$$\dot{x} = v \tag{1.8a}$$

$$\dot{v} = a - C_\mathrm{d}|v|v \tag{1.8b}$$

where x is the position of the vehicle, v its velocity and C_d the drag coefficient. For a small passenger car it would be for instance $C_\mathrm{d} \approx 7 \cdot 10^{-4}\ \mathrm{m}^{-1}$. Considering that vehicles should only drive with positive velocity and linearizing the second term of the right hand side of (1.8b) around v_0 we get the approximation

$$\ddot{x} = a - 2C_\mathrm{d}v_0 \cdot v \tag{1.9}$$

Later we would like to take into account the time delay in the braking system and engine as well as "force" a control bandwidth limitation to improve robustness by establishing $T_\mathrm{d} = 50$ ms. Using the acceleration a as the input and the position x as the output, the plant transfer function for the linearized system would then be

$$P(s) = \frac{1}{s(s + 2C_\mathrm{d}v_0)}\mathrm{e}^{-T_\mathrm{d}s} \tag{1.10}$$

A chain of N cars should drive in a row with a prescribed distance x_d between them and the first car (Index 1) is to follow a given trajectory $x_\mathrm{r}(t)$. We therefore express the control objectives as follows:

$$e_1 = x_\mathrm{r} - x_1 - x_\mathrm{d} \xrightarrow{t\to\infty} 0 \tag{1.11a}$$

$$e_i = x_{i-1} - x_i - x_\mathrm{d} \xrightarrow{t\to\infty} 0, \qquad \forall \quad 1 < i \le N \tag{1.11b}$$

where the e_i are the deviation from their desired positions.

1.4 Linear Controller Design

In order to form a homogeneous string of vehicles where every car follows its predecessor and the leading car is to follow a given trajectory every vehicle needs to be regulated by the same controller. First we want to use unity

[1] However, more detailed and complex models are discussed in [10, 35].

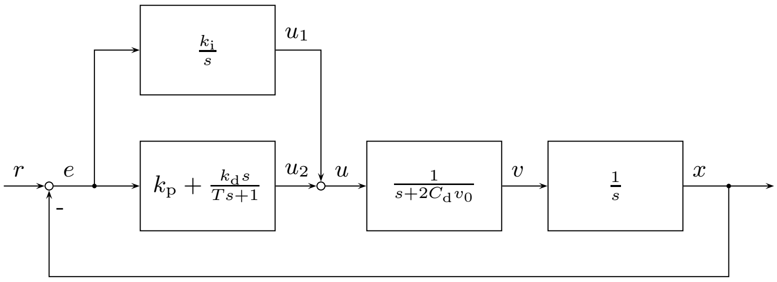


Figure 1.3: Linear system: Block diagram

feedback and design a simple PID-controller of the form

$$C(s) = \frac{k_\mathrm{i}}{s} + k_\mathrm{p} + \frac{k_\mathrm{d} s}{Ts+1} \tag{1.12a}$$

$$= K\frac{(s+z)^2}{s(s+p)} \tag{1.12b}$$

This may be implemented in a parallel form leading to the overall block diagram shown in Figure 1.3 with the following state space realisation for the controller

$$\dot{x}_{\mathrm{c}_1} = k_\mathrm{i} e \tag{1.13a}$$

$$\dot{x}_{\mathrm{c}_2} = -\frac{1}{T}x_{\mathrm{c}_2} + e \tag{1.13b}$$

$$u = x_{\mathrm{c}_1} - \frac{k_\mathrm{d}}{T^2}x_{\mathrm{c}_2} + \left(k_\mathrm{p} + \frac{k_\mathrm{d}}{T}\right)e \tag{1.13c}$$

Choosing $k_\mathrm{i} = 0.17$, $k_\mathrm{p} = 1.66$, $k_\mathrm{d} = 4.10$ and $T = 1/30$ or $K = 124.8$, $z = 0.2$ and $p = 30$, respectively, leads to an individual stable system with a fairly large phase margin of $\approx 65°$ (see Bode plot in Figure 1.4).

1.5 Simulation Results

The behaviour of the string will be tested by simulating two different manoeuvres.

In manoeuvre I all cars start with $v_{0,i} = 0$ and the desired distance $x_\mathrm{d} = 10$ m between them ($e_{i,0} = 0$). The first car should follow a ramp $x_\mathrm{r}(t) = 30\ \mathrm{m/s} \cdot t$. We observe in Figure 1.5 that every individual subsystem is stable but the string is unstable since the error is increasing with the index i (Subfigure 1.5d). Beyond a certain string length collisions would occur, see for instance (Subfigure 1.5a). In this simplified linear scenario the disturbance response is so poor that some vehicles would even have negative velocity (Subfigure 1.5b). Note also that the simulated maximum acceleration $a_\mathrm{max} = 125\ \mathrm{m/s^2}$ (see Subfigure 1.5c) is rather unrealistic for a passenger car. However, we defer discussion of this important non-linear effect to Chapter 3.

In manoeuvre II every car starts with the desired velocity $v_\mathrm{r} = 30$ m/s and the desired distance $x_\mathrm{d} = 10$ m to its predecessor. At $t = 0$ a step of $x_\mathrm{s} = 5$ m

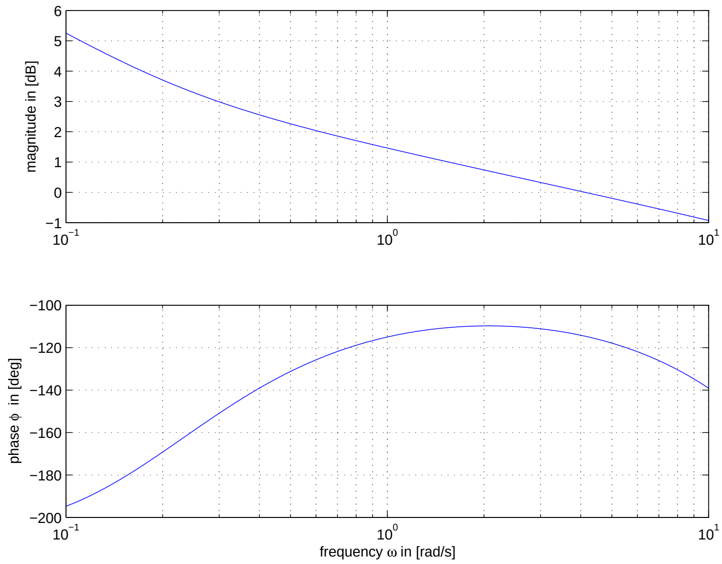
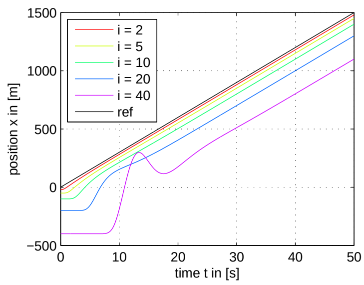
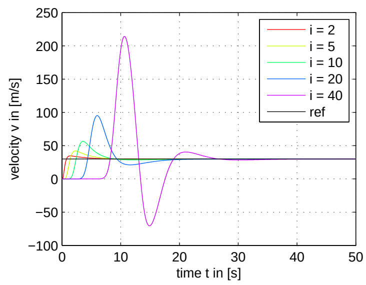


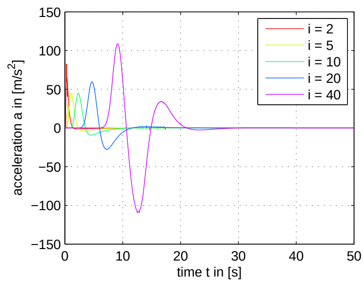
Figure 1.4: Linear system: Bode plot of the open loop system



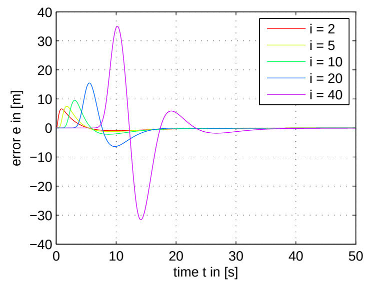
(a) Position x



(b) Velocity v

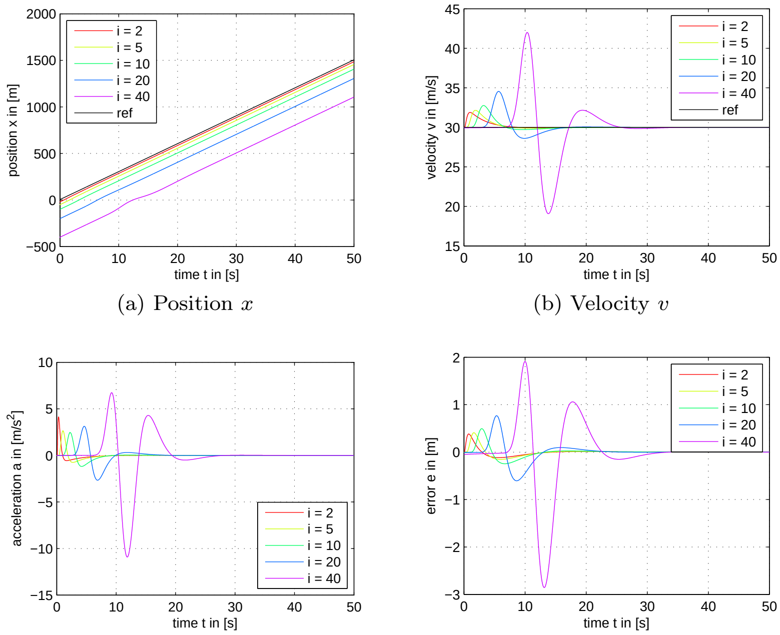


(c) Acceleration a



(d) Error e

Figure 1.5: Linear system: Simulation of manoeuvre I



(a) Position x (b) Velocity v

(c) Acceleration a (d) Error e

Figure 1.6: Linear system: Simulation of manoeuvre II

is added to the ramp $x_\mathrm{r}(t)$. The effect of this small disturbance is amplified successively by each system in the string (see Figure 1.6).

1.6 String Stability Analysis

1.6.1 Deviation Coordinates and System Dynamics

To facilitate analysis of string stability, let us take a closer look at the states of the system of Figure 1.3.

Since the position of the i^th car is regulated according to the position of the $(i-1)^\mathrm{th}$ car and since the first car follows a ramp, these states are unbounded. Thus, we need to transform the original states into deviation or error coordinates in order to discuss string stability.

First we will introduce a vector X_i describing the states of the i^th car.

$$X_i^\mathrm{T} = \begin{pmatrix} x_i & v_i & x_{\mathrm{c}1_i} & x_{\mathrm{c}2_i} \end{pmatrix} \tag{1.14}$$

where $x_{\mathrm{c}1_i}$ and $x_{\mathrm{c}2_i}$ correspond to the controller states given in (1.13a) and (1.13b). Moreover the derivative of X_i is

$$\dot{X}_i = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2C_\mathrm{d}v_0 & 1 & -\frac{k_\mathrm{d}}{T^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{T} \end{bmatrix}}_{A'} X_i + \underbrace{\begin{pmatrix} 0 \\ k_\mathrm{p} + \frac{k_\mathrm{d}}{T} \\ k_\mathrm{i} \\ 1 \end{pmatrix}}_{b} e_i \tag{1.15}$$

where

$$\begin{aligned} e_1 &= x_\mathrm{r} - x_1 - x_\mathrm{d} \\ &= c^\mathrm{T}(X_\mathrm{r} - X_1) - x_\mathrm{d} && \text{(1.16a)} \\ e_i &= x_{i-1} - x_i - x_\mathrm{d} \\ &= c^\mathrm{T}(X_{i-1} - X_i) - x_\mathrm{d} \qquad \forall 1 < i \leq N && \text{(1.16b)} \end{aligned}$$

and

$$c^\mathrm{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \tag{1.17}$$

as well as the reference vector

$$X_\mathrm{r}^\mathrm{T} = \begin{pmatrix} x_\mathrm{r} & v_\mathrm{r} & 2C_\mathrm{d} v_0 v_\mathrm{r} & 0 \end{pmatrix} \tag{1.18}$$

Furthermore we define the local state error ξ_i such that steady state is achieved when all of its elements are zero.

$$\begin{aligned} \xi_1 &= X_\mathrm{r} - X_1 - c x_\mathrm{d} && \text{(1.19a)} \\ \xi_i &= X_{i-1} - X_i - c x_\mathrm{d} \qquad \forall 1 < i \leq N && \text{(1.19b)} \end{aligned}$$

Note that $\dot{X}_\mathrm{r}$ is not zero but $\begin{pmatrix} v_\mathrm{r} & 0 & 0 & 0 \end{pmatrix}^\mathrm{T}$, which can be written as $A' X_\mathrm{r}$. Furthermore $A'c = 0$ and $c^\mathrm{T} c = 1$. Using these facts, differentiating (1.19) results in

$$\begin{aligned} \dot{\xi}_1 &= \dot{X}_\mathrm{r} - A'X_1 - bc^\mathrm{T}(X_\mathrm{r} - X_1) + b x_\mathrm{d} \\ &= A'\xi_1 + \dot{X}_\mathrm{r} - A'X_\mathrm{r} + A'c x_\mathrm{d} - bc^\mathrm{T}\xi_1 - bc^\mathrm{T} c x_\mathrm{d} + b x_\mathrm{d} \\ &= \underbrace{(A' - bc^\mathrm{T})}_{=:A} \xi_1 && \text{(1.20a)} \\ \dot{\xi}_i &= A'X_{i-1} + bc^\mathrm{T}(X_{i-2} - X_{i-1}) - b x_\mathrm{d} - A'X_i - bc^\mathrm{T}(X_{i-1} - X_i) + b x_\mathrm{d} \\ &= A'\xi_i + A'c x_\mathrm{d} + bc^\mathrm{T}\xi_{i-1} - bc^\mathrm{T}\xi_i + bc^\mathrm{T} c x_\mathrm{d} - bc^\mathrm{T} c x_\mathrm{d} \\ &= \underbrace{(A' - bc^\mathrm{T})}_{A} \xi_i + bc^\mathrm{T}\xi_{i-1} \qquad \forall 1 < i \leq N && \text{(1.20b)} \end{aligned}$$

Finally, the behavior of all error coordinates is given by

$$\underbrace{\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_N \end{pmatrix}}_{\dot{\xi}} = \underbrace{\begin{bmatrix} A & & & 0 \\ bc^\mathrm{T} & A & & \\ & \ddots & \ddots & \\ 0 & & bc^\mathrm{T} & A \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix}}_{\xi} \tag{1.21}$$

Clearly this approach could be used for heterogeneous strings as well, where the behaviour of the subsystems is not described by A, b, c^T, but rather individual, subsystem depending A_i, b_i and c_i^T.

As required by Definition 1 and Definition 2, initial conditions of all but the first subsystem must be zero for both manoeuvres. The initial conditions for the first car are

$$\xi_{1_I}(0) = \begin{pmatrix} 0 \\ v_\mathrm{r} \\ 2C_\mathrm{d} v_0 v_\mathrm{r} \\ 0 \end{pmatrix}, \qquad \xi_{1_{II}}(0) = \begin{pmatrix} x_\mathrm{s} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.22}$$

Through this procedure, we have transformed an input output stability problem into an initial condition problem and we are now able to study string stability according to Definition 1 and Definition 2.

As the maximum of all the signals in the system needs to be found, we must formally determine the solution of (1.21). Using the matrix exponential function, it can be written as

$$\begin{aligned} \xi(t) &= \mathrm{e}^{\tilde{A}t}\xi(0) \\ &= \mathcal{L}^{-1}\left\{\left(sI - \tilde{A}\right)^{-1}\right\}\xi(0) \end{aligned} \tag{1.23}$$

Thus, to calculate $\mathrm{e}^{\tilde{A}t}$ let us look at the inverse Laplace transform of the inverse of $(sI - \tilde{A})$.

$$(sI - \tilde{A})^{-1} = \begin{bmatrix} sI - A & & & \text{\Large 0} \\ -bc^\mathrm{T} & sI - A & & \\ & \ddots & \ddots & \\ \text{\Large 0} & & -bc^\mathrm{T} & sI - A \end{bmatrix}^{-1} \tag{1.24}$$

which can be written elementwise as

$$(sI - \tilde{A})^{-1}_{ij} = \begin{cases} 0 & \text{for } i < j \\ (sI - A)^{-1} & \text{for } i = j \\ (sI - A)^{-1} b \Gamma^{(i-j-1)}(s) c^\mathrm{T} (sI - A)^{-1} & \text{for } i > j \end{cases} \tag{1.25}$$

where $\Gamma(s) = c^\mathrm{T}(sI - A)^{-1}b$ is the complementary sensitivity function $T(s)$ of the feedback loop in Figure 1.3 on page 8.

1.6.2 L_2-String Stability

Since we have observed in Section 1.5 on Figure 1.5 and Figure 1.6 that the error e_i is growing with i, we do not expect the system to be L_2- or even L_∞-string stable.

Therefore we will show that there exist initial conditions such that the norm of the error signal e_N is growing with N to demonstrate that the system does not satisfy Definition 1 or Definition 2.

As $\xi_i(0)$ is zero for all $i > 1$, we can write

$$\begin{aligned}
\xi(t) &= \mathcal{L}^{-1}\{(sI - \tilde{A})^{-1}\}\xi(0) \\
&= \begin{bmatrix} \mathcal{L}^{-1}\{(sI - A)^{-1}\}\xi_1(0) \\ \mathcal{L}^{-1}\{(sI - A)^{-1}bc^{\mathrm{T}}(sI - A)^{-1}\}\xi_1(0) \\ \vdots \\ \mathcal{L}^{-1}\{(sI - A)^{-1}bT^{N-2}(s)c^{\mathrm{T}}(sI - A)^{-1}\}\xi_1(0) \end{bmatrix}
\end{aligned} \tag{1.26}$$

In order to analyse $e_N(t)$, take $\xi_1(0) = b$ and consider $e_N(t) = c^{\mathrm{T}}\xi_N(t)$. Then we can rewrite the last row of (1.26) as

$$\begin{aligned}
e_N(t) &= \mathcal{L}^{-1}\{c^{\mathrm{T}}(sI - A)^{-1}bT^{N-2}(s)c^{\mathrm{T}}(sI - A)^{-1}b\} \\
&= \mathcal{L}^{-1}\{T^N(s)\}
\end{aligned} \tag{1.27}$$

In other words the behaviour of $e_N(t)$ is fully specified by $T(s)$.

Middleton showed in [19] how performance of linear systems with unit feedback is limited: Let $L(s) = L_0(s) \cdot \mathrm{e}^{-T_{\mathrm{d}}s}$ with $L_0(s)$ being a strictly proper rational function with a set of non minimal phase zeros q_i and $T_{\mathrm{d}} > 0$ and let $T(s) = L(s)/(1 + L(s))$ be the complementary sensitivity function. Then, if the system is stable, $T(s)$ satisfies

$$\int_0^\infty \log\left|\frac{T(j\omega)}{T(0)}\right| \frac{\mathrm{d}\omega}{\omega^2} = \frac{\pi}{2}\frac{1}{T(0)}\lim_{s\to 0}\frac{\mathrm{d}T(s)}{\mathrm{d}s} + \pi\sum_i \frac{1}{q_i} + \frac{\pi}{2}T_{\mathrm{d}} \tag{1.28}$$

There are no non-minimum phase zeros in our system and therefore $\sum_i \frac{1}{q_i} = 0$. Since both plant and controller provide a pole at the origin, $\lim_{s\to 0}\frac{\mathrm{d}T(s)}{\mathrm{d}s} = 0$. Furthermore $T(0) = 1$ and thus

$$\int_0^\infty \log|T(j\omega)| \frac{\mathrm{d}\omega}{\omega^2} = \frac{\pi}{2}T_{\mathrm{d}} \tag{1.29}$$

This implies that there exists a frequency ω_0 where $|T(j\omega_0)| > 1$. Since $T(j\omega)$ is a continuous function of ω, there exist an interval $[\omega_1,\omega_2]$ and a $\epsilon > 0$, such that $|T(j\omega)| \geq 1 + \epsilon$ for all $\omega \in [\omega_1,\omega_2]$.

Using Parseval's inequality and (1.27), we can calculate the square of the L_2-norm of $e_N(t)$ for the initial condition given

$$\begin{aligned}
||e_N(\,\cdot\,)||_{L_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |e_N(j\omega)|^2 \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |T^N(j\omega)|^2 \mathrm{d}\omega
\end{aligned} \tag{1.30}$$

Since

$$\int_R f(x)\mathrm{d}x \geq \int_{T\subset R} f(x)\mathrm{d}x \tag{1.31}$$

for any non-negative function $f(x)$, we can bound (1.30) by

$$
\begin{aligned}
||e_N(\,\cdot\,)||_{L_2}^2 &\geq \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} |T^N(j\omega)|^2 \mathrm{d}\omega \\
&\geq \frac{1}{2\pi}(\omega_2-\omega_1)(1+\epsilon)^{2N} \qquad (1.32)
\end{aligned}
$$

So $||e_N(\,\cdot\,)||_{L_2}$ will grow as N grows. Since $||\xi(\,\cdot\,)||_{L_2} \geq ||\xi_N(\,\cdot\,)||_{L_2} \geq ||e_N(\,\cdot\,)||_{L_2}$ it follows that the system cannot be L_2-string stable in the sense of Definition 2.

1.6.3 L_∞-String Stability

To close this section, we will show, that the system cannot be L_∞-string stable either by expanding the argument above. Note that since $T(s)$ is analytic in the closed right half plane, it is continuous there. Therefore, there exist a $\eta > 0$ and a $\epsilon_\eta \in (0,\epsilon)$, such that $|T(\eta + j\omega)| \geq 1 + \epsilon_\eta$ for all $\omega \in [\omega_1,\omega_2]$. Therefore the following analogy to (1.30) and (1.32) holds

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} |e_N(\eta+j\omega)|^2 \mathrm{d}\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |T(\eta+j\omega)|^2 N \mathrm{d}\omega \\
&\geq \frac{1}{2\pi}(\omega_2-\omega_1)(1+\epsilon_\eta)^{2N} \qquad (1.33)
\end{aligned}
$$

Using Parseval's Theorem and frequency shifting, we obtain

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} |e_N(\eta+j\omega)|^2 \mathrm{d}\omega &= \int_{0}^{\infty} e_N^2(t)\mathrm{e}^{-2\eta t} \mathrm{d}t \\
&\leq \frac{1}{2\eta}||e_N(\,\cdot\,)||_{L_\infty}^2 \qquad (1.34)
\end{aligned}
$$

Combining (1.33) and (1.34), we see that

$$
||e_N(\,\cdot\,)||_{L_\infty} \geq (1+\epsilon_\eta)^N \sqrt{\frac{\eta}{\pi}(\omega_2-\omega_1)} \qquad (1.35)
$$

and therefore the system cannot be L_∞-string stable according to Definition 1.

Thus for this class of problems considered here it is not possible to obtain L_2- or L_∞-string stability using linear, time invariant, unit feedback controllers that include integral action (see [5, 7, 20, 24, 26, 31]).

Therefore we will now seek for other approaches to control the vehicles in order to obtain string stability such as using a velocity depending spacing policy in Chapter 2 and nonlinear controllers in Chapter 3.

Chapter 2

Linear Controller with Time Headway

Having discussed the inability of unit feedback linear controllers to guarantee string stability, we shall now propose a suitable extension. We will show that a sufficiently large velocity dependent target spacing between the vehicles guarantees both L_2- and L_∞-string stability.

2.1 Introduction

As we have seen in the previous chapter we can obtain neither L_2- nor L_∞-string stability if a constant distance between the vehicles within the string is required. Chien and Ioannou therefore introduced in [4] a *time headway*, i.e. a velocity depending spacing policy.

In this chapter we will require a combination of a fixed and a linear velocity depending distance between the cars. We will realise minor changes in the introduced PID-controller in order to maintain the closed loop poles achieved by the PID-controller in Chapter 1.

Again we will transform the system into deviation coordinates to analyse the behaviour of the string. We will derive sufficient time headways to guarantee string stability according to Definition 1 and Definition 2 and show that not only the local error states e_i but all deviation states satisfy the requirements in the string stability definitions.

In the last section we will use the manoeuvres introduced above and simulate the string using the derived values for the minimum time headways to guarantee L_2- and L_∞-string stability.

2.2 Linear Controller Design with Time Headway

Since we have seen that string stability is unavoidable using a fixed spacing policy we will change the local error states e_i to

$$e_1 = x_\mathrm{r} - x_1 - x_\mathrm{d} - hv_1 \xrightarrow{t\to\infty} 0 \tag{2.1a}$$

$$e_i = x_{i-1} - x_i - x_\mathrm{d} - hv_i \xrightarrow{t\to\infty} 0, \qquad \forall \quad 1 < i \leq N \tag{2.1b}$$

with the time headway h. Thus, a vehicle driving with a high velocity will keep a larger distance to its predecessor.

If $Q(s) = hs + 1$ the transfer function of the new closed loop system describing how the position of the i^{th} car is following the trajectory of x_{i-1} or x_{r}, respectively, will be

$$\Gamma(s) = \frac{C_{\text{h}}(s)P(s)}{1 + C_{\text{h}}(s)P(s)Q(s)} \tag{2.2}$$

Since having the same closed loop poles as above will guarantee the same stability characteristics for the individual system we will choose $C_{\text{h}}(s) = C(s)/Q(s)$ with $C(s)$ as in (1.12) on page 8 and obtain

$$\Gamma(s) = \frac{1}{Q(s)} \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{1}{Q(s)} T(s) \tag{2.3}$$

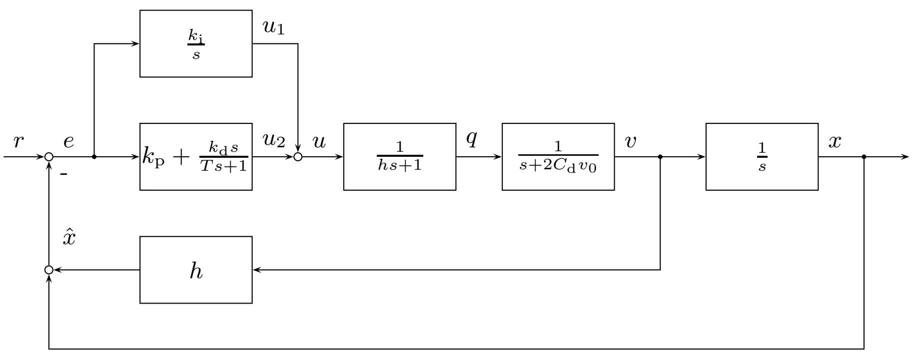


Figure 2.1: Linear system with time headway: Block diagram

2.3 String Stability Analysis

2.3.1 Deviation Coordinates and System Dynamics

To transform the system into error coordinates we first describe it using the states of the system in Figure 2.1. Since neither $x_{i-1} - x_i$ nor $\hat{x}_{i-1} - \hat{x}_i$ go to zero but the local error $e_i = x_{i-1} - x_i - hv_i - x_{\text{d}}$ disappears as t goes to infinity we will consider e_i as the first state of the i^{th} state vector X.

$$X_i^{\text{T}} = \begin{pmatrix} e_i & v_i & q_i & x_{\text{c}_{1_i}} & x_{\text{c}_{2_i}} \end{pmatrix} \tag{2.4}$$

Its derivative is

$$\dot{X}_i = \underbrace{\begin{bmatrix} 0 & -1 + h2C_{\text{d}}v_0 & -h & 0 & 0 \\ 0 & -2C_{\text{d}}v_0 & 1 & 0 & 0 \\ \frac{1}{h}\left(k_{\text{p}} + \frac{k_{\text{d}}}{T}\right) & 0 & -\frac{1}{h} & \frac{1}{h} & -\frac{1}{h}\frac{k_{\text{d}}}{T^2} \\ k_{\text{i}} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -\frac{1}{T} \end{bmatrix}}_{A_{\text{h}}} X_i + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b_{\text{h}}} v_{i-1} \tag{2.5}$$

with

$$v_i = \underbrace{\begin{pmatrix}0 & 1 & 0 & 0 & 0\end{pmatrix}}_{c_{\mathrm{h}}^{\mathrm{T}}} X_i \tag{2.6}$$

We simply define the error states ξ_i as

$$\xi_1 = X_{\mathrm{r}} - X_1 \tag{2.7a}$$
$$\xi_i = X_{i-1} - X_i \qquad \forall 1 < i \leq N \tag{2.7b}$$

with the reference vector

$$X_{\mathrm{r}}^{\mathrm{T}} = \begin{pmatrix}0 & v_{\mathrm{r}} & 2C_{\mathrm{d}}v_0v_{\mathrm{r}} & 2C_{\mathrm{d}}v_0v_{\mathrm{r}} & 0\end{pmatrix} \tag{2.8}$$

and the error vector derivative

$$\begin{aligned}\dot{\xi}_1 &= -A_{\mathrm{h}}X_1 - b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}}X_{\mathrm{r}} \\ &= A_{\mathrm{h}}\xi_1 - A_{\mathrm{h}}X_{\mathrm{r}} - b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}}X_{\mathrm{r}} \\ &= A_{\mathrm{h}}\xi_1 \end{aligned} \tag{2.9a}$$
$$\begin{aligned}\dot{\xi}_i &= A_{\mathrm{h}}X_{i-1} + b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}}X_{i-2} - A_{\mathrm{h}}X_i - b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}}X_{i-1} \\ &= A_{\mathrm{h}}\xi_i + b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}}\xi_{i-1} \qquad \forall 1 < i \leq N \end{aligned} \tag{2.9b}$$

Note that $A_{\mathrm{h}}X_{\mathrm{r}} = -b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}}X_{\mathrm{r}}$ and that X_{r} does not contain x_{r} anymore and its derivative therefore is zero. Again, the behavior of all states can be described by

$$\underbrace{\begin{pmatrix}\dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_N\end{pmatrix}}_{\dot{\xi}} = \underbrace{\begin{bmatrix}A_{\mathrm{h}} & & & 0 \\ b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}} & A_{\mathrm{h}} & & \\ & \ddots & \ddots & \\ 0 & & b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}} & A_{\mathrm{h}}\end{bmatrix}}_{\tilde{A_{\mathrm{h}}}} \underbrace{\begin{pmatrix}\xi_1 \\ \xi_2 \\ \vdots \\ \xi_N\end{pmatrix}}_{\xi} \tag{2.10}$$

The initial conditions for the first subsystem for manoeuvre I and II described above now are

$$\xi_{1_I} = \begin{pmatrix}0 \\ v_{\mathrm{r}} \\ 2C_{\mathrm{d}}v_0v_{\mathrm{r}} \\ 2C_{\mathrm{d}}v_0v_{\mathrm{r}} \\ 0\end{pmatrix}, \qquad \xi_{1_{II}} = \begin{pmatrix}-x_{\mathrm{s}} \\ 0 \\ 0 \\ 0 \\ 0\end{pmatrix} \tag{2.11}$$

while the initial values of ξ_i are zero for all following vehicles in both manoeuvres.

As shown in Section 1.6 on page 12 and below the solution of the system in (2.10) can be derived building the inverse Laplace transform of $(sI - \tilde{A}_{\mathrm{h}})^{-1}$. Note that instead of A, b and c^{T} now A_{h}, b_{h} and $c_{\mathrm{h}}^{\mathrm{T}}$ are of interest. Thus, $\Gamma(s) = c_{\mathrm{h}}^{\mathrm{T}}(sI - A_{\mathrm{h}})^{-1}b_{\mathrm{h}}$ is not the former complementary sensitivity function $T(s)$ but $\Gamma(s) = Q(s)^{-1}T(s)$ with $Q = hs + 1$ (as considered in (2.3) on page 16).

We wish to look back on (1.26) on page 13. We will substitute the new complementary sensitivity function $\Gamma(s)$ for $T(s)$ and A, b and c^{T} with A_{h}, b_{h} and $c_{\mathrm{h}}^{\mathrm{T}}$.

$$\begin{aligned}
\xi(t) &= \mathcal{L}^{-1}\{(sI - \tilde{A}_{\mathrm{h}})^{-1}\}\xi(0) \\
&= \begin{bmatrix} \mathcal{L}^{-1}\{(sI - A_{\mathrm{h}})^{-1}\}\xi_1(0) \\ \mathcal{L}^{-1}\{(sI - A_{\mathrm{h}})^{-1} b_{\mathrm{h}} c_{\mathrm{h}}^{\mathrm{T}} (sI - A_{\mathrm{h}})^{-1}\}\xi_1(0) \\ \vdots \\ \mathcal{L}^{-1}\{(sI - A_{\mathrm{h}})^{-1} b_{\mathrm{h}} \Gamma^{N-2}(s) c_{\mathrm{h}}^{\mathrm{T}} (sI - A_{\mathrm{h}})^{-1}\}\xi_1(0) \end{bmatrix}
\end{aligned}$$

The product of inverse Laplce transforms can be written as the convolution of the corresponding impulse responses: $\alpha(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$, $\phi(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}b\}$, and $\gamma(t) = \mathcal{L}^{-1}\{\Gamma(s)\}$.

$$\xi(t) = \begin{bmatrix} \delta(t) * \alpha(t) \\ \phi(t) * c\alpha(t) \\ \phi(t) * \gamma(t) * c\alpha(t) \\ \vdots \\ \phi(t) * \gamma(t)^{*(N-2)} * c\alpha(t) \end{bmatrix} \xi_1(0)$$

Note that $\gamma(t)^{*i} = \underbrace{\gamma(t) * \gamma(t) * \ldots * \gamma(t)}_{i \text{ times}}$.

2.3.2 L_∞-String Stability

In order to satisfy L_∞-string stability according to Definition 1 all states in $\xi(t)$ need to be bounded.

$$\begin{aligned}
||\xi(\,\cdot\,)||_{L_\infty} &= \left\| \begin{bmatrix} \delta(t) * \alpha(t) \\ \phi(t) * c\alpha(t) \\ \phi(t) * \gamma(t) * c\alpha(t) \\ \vdots \\ \phi(t) * \gamma(t)^{*(N-2)} * c\alpha(t) \end{bmatrix} \xi_1(0) \right\|_{L_\infty} \\
&\leq \left\| \begin{bmatrix} \delta(t) * \alpha(t) \\ \phi(t) * c\alpha(t) \\ \phi(t) * \gamma(t) * c\alpha(t) \\ \vdots \\ \phi(t) * \gamma(t)^{*(N-2)} * c\alpha(t) \end{bmatrix} \right\|_{\mathrm{i}_\infty} ||\xi_1(0)||_\infty \\
&\leq \left\| \begin{bmatrix} \delta(t) \\ \phi(t) \\ \phi(t) * \gamma(t) \\ \vdots \\ \phi(t) * \gamma(t)^{*(N-2)} \end{bmatrix} \right\|_{\mathrm{i}_\infty} ||\alpha(t)||_{L_\infty} ||\xi_1(0)||_\infty \qquad (2.12)
\end{aligned}$$

since $|c| = 1$. Here the induced norm is the maximal row sum. That leads us to the maximal entry of the block matrix. Since the controller is designed in a way that A is Hurwitz the impulse responses $\alpha(t)$ and $\phi(t)$ are bounded. Thus the maximal row sum will depend on the norm of $\gamma^{*(N-2)}(t)$ which is always less or equal to the norm of $\gamma(t)$ to the power of $N-2$ ($||\gamma^{*(N-2)}(t)|| \leq ||\gamma(t)||^{N-2}$).

Note that the induced ∞-norm of an operator is the L_1-norm of its impulse response. Hence, one way to insure that the system is L_∞-string stable is to require $||\gamma(t)||_{L_1}$ to be less or equal than 1. (In fact $||\gamma(t)||_{L_1}$ cannot be less than 1 in this example. Thus $||\gamma(t)||_{L_1} = 1$ is required.) That is the same as requiring a monotonically non-decreasing step response or equivalently a non negative impulse response.

Since the complementary sensitivity function has changed from $T(s)$ to $\Gamma(s) = Q(s)^{-1}T(s)$ the corresponding impulse response changed as well from $\gamma_0(t) = \mathcal{L}^{-1}\{T(s)\}$ to $\gamma(t) = \mathcal{L}^{-1}\{\Gamma(s)\}$.

$$\begin{aligned}
\gamma(t) &= \mathcal{L}^{-1}\left\{\frac{1}{hs+1}T(s)\right\} \\
&= \frac{1}{h}\mathrm{e}^{-\frac{t}{h}} * \gamma_0(t) \\
&= \int_0^t \frac{1}{h}\mathrm{e}^{-\frac{t-\tau}{h}}\gamma_0(\tau)\mathrm{d}\tau \\
&= \frac{1}{h}\mathrm{e}^{-\frac{t}{h}}\int_0^t \mathrm{e}^{\frac{\tau}{h}}\gamma_0(\tau)\mathrm{d}\tau
\end{aligned} \tag{2.13}$$

Since $\frac{1}{h}e^{-\frac{t}{h}} > 0$ the integral over $e^{\frac{t}{h}}\gamma_0(t)$ needs to be greater or equal then zero.

$$\gamma(t) = \frac{1}{h}\mathrm{e}^{-\frac{t}{h}}\int_0^t \mathrm{e}^{\frac{\tau}{h}}\gamma_0(\tau)\mathrm{d}\tau \geq 0 \quad \Longleftrightarrow \quad \underbrace{\int_0^t \mathrm{e}^{\frac{\tau}{h}}\gamma_0(\tau)\mathrm{d}\tau}_{\bar{\gamma}(t)} \geq 0 \tag{2.14}$$

If possible we need to find a h_∞ which is the minimal time headway h satisfying (2.14). Thus, every $h \geq h_\infty$ guarantees L_∞-string stability.

Since $\gamma(t)$ is continuous we only need to make sure that all local maxima and minima of $\bar{\gamma}(t)$ are greater or equal then zero. Local extrema appear if the derivative of $\bar{\gamma}(t)$ is zero

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\gamma}(t)) = e^{\frac{t}{h}}\gamma_0(t) = 0 \quad \Longleftrightarrow \quad \gamma_0(t) = 0 \tag{2.15}$$

Thus, we must assure that $\gamma_0(t)$ changes sign only a finite number of times by requiring that the dominant pole of $T(s)$ is real. Then $\bar{\gamma}(t)$ must be non negative at the zero crossings of $\gamma_0(t)$. If $\bar{\gamma}(t)$ is non negative at all zero crossings of $\gamma_0(t)$ the impulse response $\gamma(t)$ will be non negative for all $t > 0$. Therefore we could guarantee L_∞-string stability.

Swaroop *et al.* established a different result focusing on the poles and zeros of $T(s)$ to assure a positive impulse response in [32].

The impulse response $\gamma_0(t)$ of the system above changes sign at $t_1 = 0.9$ s and $t_2 = 15.5$ s. We need to find the minimal time headway h_∞ to assure that

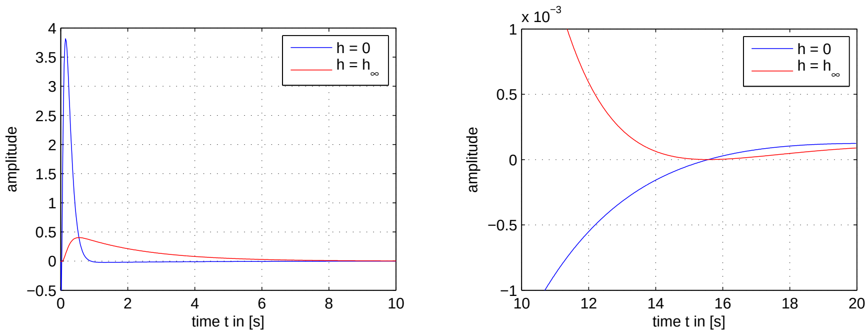


Figure 2.2: Linear system with time headway: Impulse responses

$\bar{\gamma}(t_1)$ and $\bar{\gamma}(t_2)$ are non negative. In Figure 2.2 we can observe that choosing $h_\infty = 2.238\,\mathrm{s}$ the impulse response $\gamma(t)$ is greater than zero at t_1 and zero at t_2. Thus using a time headway greater or equal than h_∞ will lead to a monotonically non-decreasing step response and a L_∞-string stable system according to Definition 1.

2.3.3 L_2-String Stability

If L_∞-string stability is not needed it is possible to obtain L_2-string stability with a smaller time headway. To satisfy Definition 2 the L_2-norm of ξ_i must not grow without bound for every i. To study this problem we go back to (1.26) on page 13 and substitute the new complementary sensitivity function $\Gamma(s)$ for $T(s)$ and A, b and c^{T} by A_{h}, b_{h} and $c_{\mathrm{h}}^{\mathrm{T}}$

$$\begin{aligned}\xi(t) &= \mathcal{L}^{-1}\{(sI-\tilde{A}_{\mathrm{h}})^{-1}\}\xi(0)\\ &= \begin{bmatrix} \mathcal{L}^{-1}\{(sI-A_{\mathrm{h}})^{-1}\}\xi_1(0) \\ \mathcal{L}^{-1}\{(sI-A_{\mathrm{h}})^{-1}b_{\mathrm{h}}c_{\mathrm{h}}^{\mathrm{T}}(sI-A_{\mathrm{h}})^{-1}\}\xi_1(0) \\ \vdots \\ \mathcal{L}^{-1}\{(sI-A_{\mathrm{h}})^{-1}b_{\mathrm{h}}\Gamma^{N-2}(s)c_{\mathrm{h}}^{\mathrm{T}}(sI-A_{\mathrm{h}})^{-1}\}\xi_1(0) \end{bmatrix}\end{aligned}$$

Note that we can use Parseval's Theorem and analyse the H_2-norm in frequency domain instead of the L_2-norm in time domain.

$$\begin{aligned}||\xi_i||_{L_2} &= ||\mathcal{L}^{-1}\left\{(sI-A_{\mathrm{h}})^{-1}b_{\mathrm{h}}\Gamma^{i-2}(s)c_{\mathrm{h}}^{\mathrm{T}}(sI-A_{\mathrm{h}})^{-1}\xi_1(0)\right\}||_{L_2}\\ &\leq ||(sI-A_{\mathrm{h}})^{-1}b_{\mathrm{h}}\Gamma^{i-2}(s)c_{\mathrm{h}}^{\mathrm{T}}(sI-A_{\mathrm{h}})^{-1}||_{H_2}\cdot||\xi_1(0)||_2\\ &\leq ||(sI-A_{\mathrm{h}})^{-1}b_{\mathrm{h}}||_{H_\infty}\cdot||\Gamma(s)||_{H_\infty}^{i-2}\cdot||c_{\mathrm{h}}^{\mathrm{T}}(sI-A_{\mathrm{h}})^{-1}||_{H_2}\cdot||\xi_1(0)||_2\end{aligned} \tag{2.16}$$

Since A_{h} is Hurwitz the values of $||(sI-A_{\mathrm{h}})^{-1}b_{\mathrm{h}}||_{H_\infty}$ and $||c_{\mathrm{h}}^{\mathrm{T}}(sI-A_{\mathrm{h}})^{-1}||_{H_2}$ are some finite numbers. To assure that $||\xi_i||_{L_2}$ is bounded the H_∞-norm of $\Gamma(s)$ has to be less or equal than 1. In fact $\Gamma(0) = 1$ and therefore we can only obtain $||\Gamma(s)||_{H_\infty} = 1$. To guarantee L_2-string stability h has to be greater or equal than h_2 which is the minimal time headway forcing $|\Gamma(s)|$ to be less or equal than 1 for all frequencies.

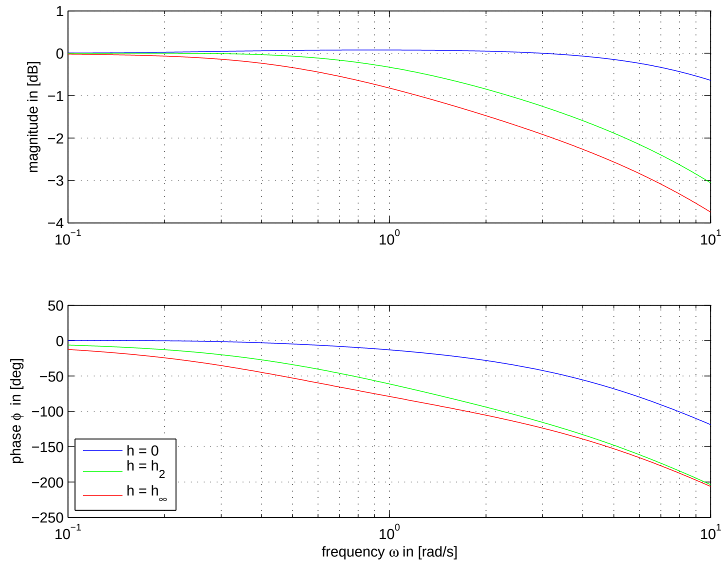


Figure 2.3: Linear system with time headway: Bode plot of the closed loop

$$|\Gamma(j\omega)|^2 = \frac{1}{1+\omega^2 h_2^2} |T(j\omega)|^2 \leq 1 \quad \forall\omega \tag{2.17}$$

$$h_2 := \sqrt{\max_{\omega} \left(\frac{|T(j\omega)|^2 - 1}{\omega^2} \right)} \tag{2.18}$$

For the example given the minimum time headway to guarantee L_2-string stability h_2 is 1.18 s. Figure 2.3 shows bode plots of the closed loop transfer function $\Gamma(j\omega)$ for different time headways. While there exist frequencies for which $|T(j\omega)|$ (complementary sensitivity function without time headway) is larger than 1, $|\Gamma(j\omega)|$ is always less or equal to 1 using a time headway of h_2 or more.

The corresponding step responses are shown in Figure 2.4. Choosing h_2 helps to reduce the overshoot significantly compared to $h = 0$. However, only choosing a time headway of h_∞ or greater will yield a monotonically non-decreasing step response.

Note that under some mild conditions $h > h_2$ is sufficient for L_∞-string stability. Consider

$$\xi_i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Xi_i(j\omega) \mathrm{e}^{-j\omega t} \mathrm{d}\omega \tag{2.19}$$

where $\Xi_i(s)$ is the Laplace transform of $\xi_i(t)$. (Since $\xi_i(t)$ goes to zero exponentially fast, the imaginary axis is in the region of absolute convergence of the Laplace transform.)

$$\Xi_i(s) = \mathcal{L}\{\xi_i(t)\} = (sI - A_\mathrm{h})^{-1} b_\mathrm{h} \Gamma^{i-2}(s) c_\mathrm{h}^\mathrm{T} (sI - A_\mathrm{h})^{-1} \xi_1(0) \tag{2.20}$$

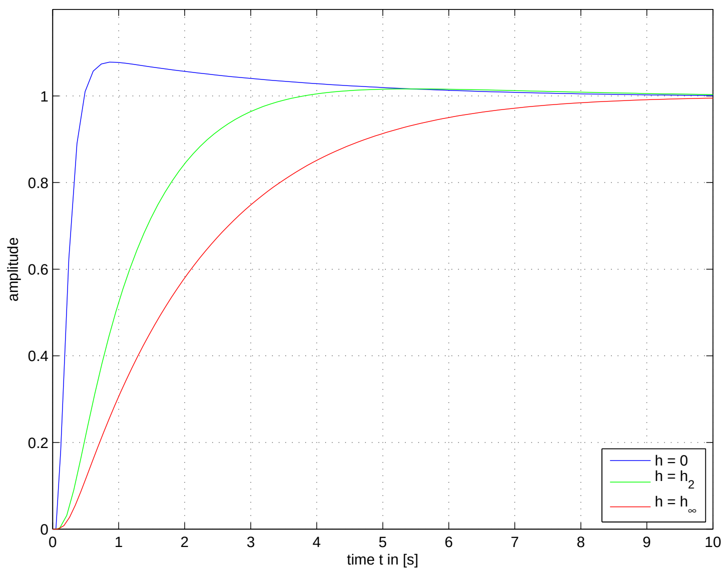


Figure 2.4: Linear system with time headway: Step responses

Suppose

$$\sup_{\omega}\left\{||(j\omega I - A_{\mathrm{h}})^{-1}||_{\mathrm{i}_\infty}\right\} \le \delta_1 \tag{2.21a}$$

$$\sup_{\omega}|\Gamma(j\omega)| \le 1 \quad \text{and} \tag{2.21b}$$

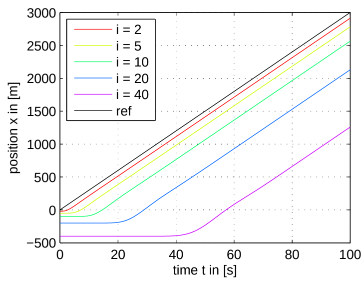
$$|\Gamma(j\omega)| \le \frac{\omega_{\mathrm{H}}}{\omega} \quad \forall|\omega| \ge |\omega_{\mathrm{H}}| \tag{2.21c}$$

$$||c_{\mathrm{h}}^{\mathrm{T}}|| = \delta_2 \tag{2.21d}$$

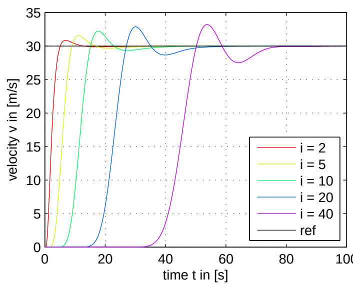
and note that $||b_{\mathrm{h}}|| = 1$. Then from (2.19) for all t

$$\begin{aligned} ||\xi_i(t)||_\infty &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} ||\Xi(j\omega)||_\infty \mathrm{d}\omega \\ &\le \frac{\delta_1^2\delta_2}{2\pi}||\xi_1(0)||_\infty \left(\int_{-\infty}^{\infty} |\Gamma(j\omega)|^{i-2}\mathrm{d}\omega\right) \\ &\le \frac{\delta_1^2\delta_2}{2\pi}||\xi_1(0)||_\infty \left(\int_{-\infty}^{-\omega_{\mathrm{H}}} \left|\frac{\omega_{\mathrm{H}}}{\omega}\right|^{i-2}\mathrm{d}\omega + \int_{-\omega_{\mathrm{H}}}^{\omega_{\mathrm{H}}} 1\mathrm{d}\omega + \int_{\omega_{\mathrm{H}}}^{\infty} \left|\frac{\omega_{\mathrm{H}}}{\omega}\right|^{i-2}\mathrm{d}\omega\right) \end{aligned} \tag{2.22}$$

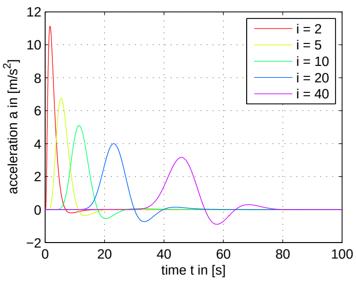
Note that all terms at the right hand side of (2.22) are bounded independent of the the position within the string i or the string length N. Note further that this is only true for $i \ge 4$. However, the first three entries in $\xi(t)$ are bounded as well since A is Hurwitz and b_{h} and $c_{\mathrm{h}}^{\mathrm{T}}$ are bounded. Hence, $\xi_i(t)$ is bounded for all t and satisfies both definitions for string stability given.



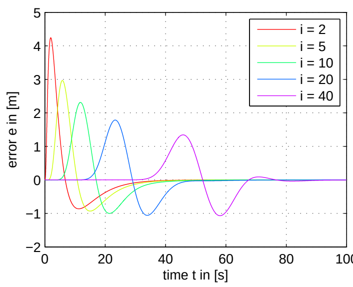
(a) Position x



(b) Velocity v



(c) Acceleration a



(d) Error e

Figure 2.5: Linear system with $h = h_2$: Simulation of manoeuvre I

2.4 Simulation Results

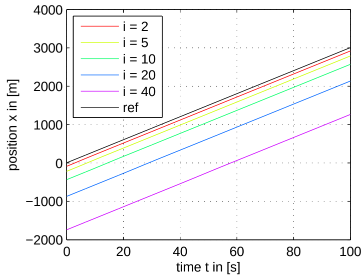
Both manoeuvres introduced above have been simulated using h_2 and h_∞.

In Figure 2.5 and Figure 2.6 we can observe that using a time headway of h_2 still causes negative error signals and overshooting velocity signals. However, the minimal distance between vehicles does not decrease with the string length N and is always larger than ≈ 9 m to prevent collisions.

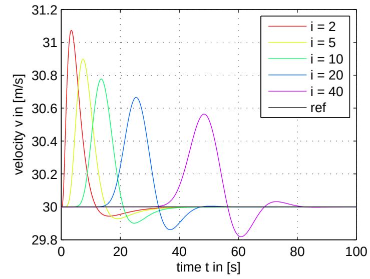
In steady state every vehicle keeps a distance of $h_2 v_\mathrm{r} + x_\mathrm{d} = 45.4$ m to its predecessor which is still too much to save energy avoiding air friction by driving in platoons. Moreover, it takes about three times longer to force all errors to zero compared with the simulations without time headway (see Figure 1.5 and Figure 1.6 on page 23).

Since the maximal values of every signal are bounded h_2 also satisfies L_∞-string stability.

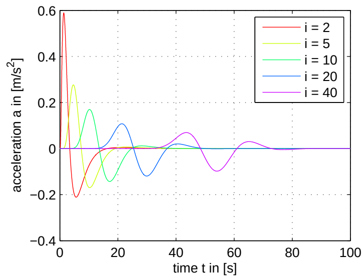
Nevertheless using a larger time headway of h_∞ provides some advantages compared to h_2. As displayed in Figure 2.7 vehicles would not brake but only accelerate while the platoon is following the ramp. This might be more energy efficient than the behaviour shown for a smaller time headway. At the same time distances between vehicles driving in steady state is very large (77.1 m) and it needs more than 100 s to force all errors to zero in both manoeuvres.



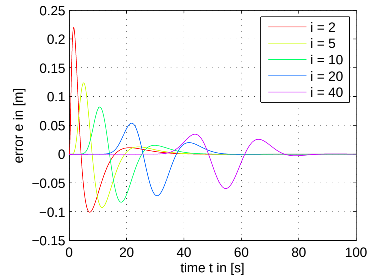
(a) Position x



(b) Velocity v

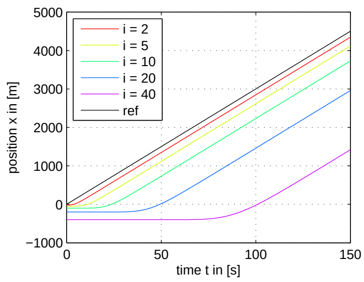


(c) Acceleration a

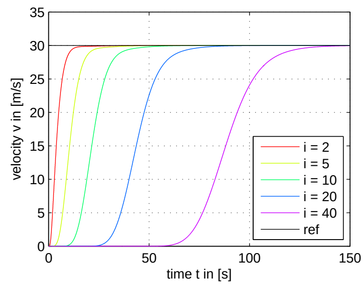


(d) Error e

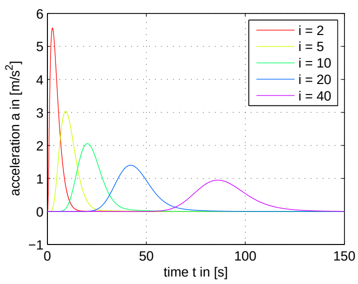
Figure 2.6: Linear system with $h = h_2$: Simulation of manoeuvre II



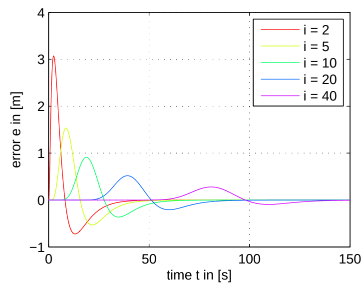
(a) Position x



(b) Velocity v

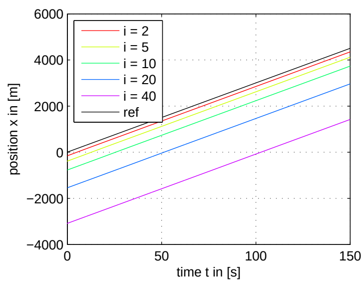


(c) Acceleration a

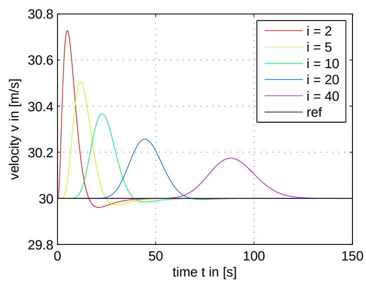


(d) Error e

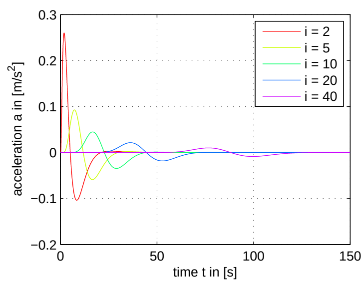
Figure 2.7: Linear system with $h = h_\infty$: Simulation of manoeuvre I



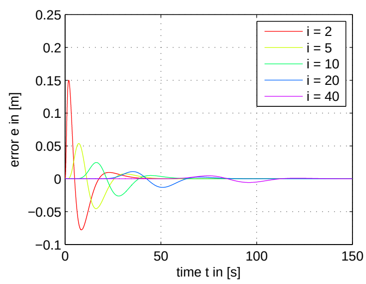
(a) Position x



(b) Velocity v



(c) Acceleration a



(d) Error e

Figure 2.8: Linear system with $h = h_\infty$: Simulation of manoeuvre II

CHAPTER 3

Nonlinear Controller

Since linear controllers with unit feedback cannot solve the string instability problem and introducing a time headway leads to slow dynamics and larger steady state separation some nonlinear controller will be implemented, tested and analysed.

3.1 Introduction

In Chapter 1 we showed that we cannot achieve any form of string stability using a linear controller with a fixed spacing policy. Indeed in Chapter 2 we could show how both L_2- and L_∞-string stability can be achieved using a sufficiently large time headway. However, this solution provoked other undesirable effects such as large steady state separation between the vehicles and comparatively slow system dynamics.

In this chapter we will apply two different nonlinear controllers, i.e. an anti-windup scheme and a controller with a signed quadratic term.

We will use the circle criterion to guarantee individual loop stability for the anti-windup scheme. Unfortunately we will observe in simulations that the anti-windup approach cannot guarantee string stability according to the definitions given above and we will give a brief explanation for this phenomenon.

In the second part of this chapter, we will focus on the PIDQ-controller, i.e. a PID-controller with a signed quadratic term. After introducing two different approaches for the PIDQ-controller we will discus stability of the subsystem, display different simulations and briefly discus string stability.

3.2 Anti-Windup

3.2.1 Anti-Windup Scheme

A passenger car can neither accelerate nor break arbitrarily rapidly. So saturating the actuating variable yields a more realistic car model.

$$u_{\text{sat}} = \begin{cases} u_{\min} & \text{for } u \leq u_{\min} \\ u & \text{for } u_{\min} < u < u_{\max} \\ u_{\max} & \text{for } u \geq u_{\max} \end{cases} \tag{3.1}$$

A bounded output of a controller with integral parts can cause problems when the actuator is saturated and the integrator keeps integrating the error signal. To avoid this behavior known as *integral windup*, several anti-windup schemes have been discussed in the literature, see for instance [6, 14] and the references therein.

In this section we shall use the scheme shown in Figure 3.1. The difference between the unsaturated and the saturated actuator signal is fed back via $H(s)$ to the controller input. Since $C(s)$ is proper but not strictly proper, $H(s)$ should be strictly proper to avoid algebraic loops in the system. All poles of $H(s)$ must have negative real parts. Furthermore we do not see any reason why $H(s)$ should provide complex poles or non-minimum phase zeros.

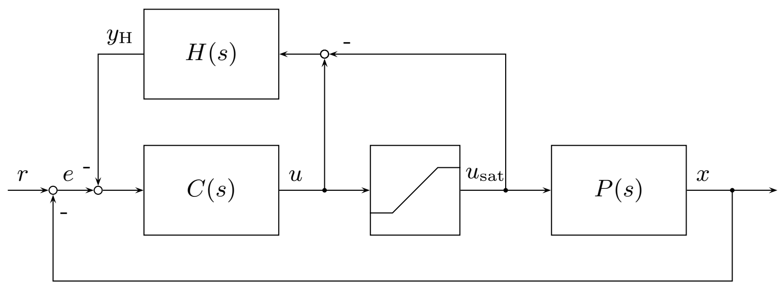


Figure 3.1: Anti-windup system: Block diagram

$$H(s) = K_{\mathrm{H}} \frac{\prod_{i=1}^{m}(s + z_{\mathrm{H}_i})}{\prod_{j=1}^{n}(s + p_{\mathrm{H}_j})}, \tag{3.2}$$

with $m < n$, $z_{\mathrm{H}_i} > 0$ and $p_{\mathrm{H}_j} > 0$. In particular, a third order approach of the form

$$H(s) = K_{\mathrm{H}} \frac{(s + z_{\mathrm{H}_1})(s + z_{\mathrm{H}_2})}{(s + p_{\mathrm{H}_1})(s + p_{\mathrm{H}_2})(s + p_{\mathrm{H}_3})} \tag{3.3}$$

has been used.

3.2.2 Stability of Anti-Windup Subsystem

Since anti-windup schemes have been widely used in many areas, stability of such systems have been studied frequently in the past decades. In the majority of cases the systems are converted into a Lur'e problem, that consists of a linear, proper transfer function $G(s)$ and a memoryless nonlinear function $\phi(y)$, as shown in Figure 3.2. Additionally, the nonlinearity $\phi(y)$ is sector $[k_1,k_2]$ bounded, i.e. $k_1 y \leq \phi(y) \leq k_2 y$. The saturation function described in (3.1) lies in sector [0,1] (see Figure 3.3). Using the circle criterion, stability of the anti-windup scheme with a stable transfer function $G(s)$ and a sector nonlinearity in [0,1] can be guaranteed, if the Nyquist curve of $G(s)$ is always right of the vertical line, $\Re(\cdot) = -1$, i.e. $\Re(G(j\omega)) > -1$ for all ω. For more details on

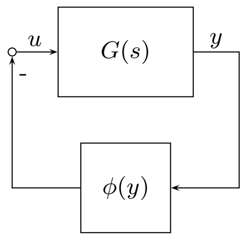


Figure 3.2: Lur'e problem: Block diagram

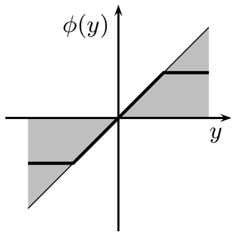


Figure 3.3: Lur'e problem: Saturation as a sector nonlinearity

Lur'e problem or the circle criterion, see e.g. [12, 34].

The critical transfer function $G(s)$ from u_{sat} to u in Figure 3.1 can be written as

$$
\begin{aligned}
G(s) =& \frac{C(s)P(s)}{1+C(s)H(s)} - \frac{C(s)H(s)}{1+C(s)H(s)} \\
=& \frac{K\frac{(s+z)^2}{s(s+p)}\frac{1}{s(s+2C_{\text{d}}v_0)}}{1+K\frac{(s+z)^2}{s(s+p)}K_{\text{H}}\frac{(s+z_{\text{H}_1})(s+z_{\text{H}_2})}{(s+p_{\text{H}_1})(s+p_{\text{H}_2})(s+p_{\text{H}_3})}} \\
&- \frac{K\frac{(s+z)^2}{s(s+p)}K_{\text{H}}\frac{(s+z_{\text{H}_1})(s+z_{\text{H}_2})}{(s+p_{\text{H}_1})(s+p_{\text{H}_2})(s+p_{\text{H}_3})}}{1+K\frac{(s+z)^2}{s(s+p)}K_{\text{H}}\frac{(s+z_{\text{H}_1})(s+z_{\text{H}_2})}{(s+p_{\text{H}_1})(s+p_{\text{H}_2})(s+p_{\text{H}_3})}}
\end{aligned}
\tag{3.4}
$$

with $C(s)$ given in pole-zero-form. Choosing $z_{\text{H}_1} = p$ and $p_{\text{H}_1} = p_{\text{H}_2} = z$ reduces the complexity of $G(s)$ significantly.

$$
G(s) = \frac{K\frac{(s+z)^2}{s(s+p)}\frac{1}{s(s+2C_{\text{d}}v_0)}}{1+KK_{\text{H}}\frac{(s+z_{\text{H}_2})}{s(s+p_{\text{H}_3})}} - \frac{KK_{\text{H}}\frac{(s+z_{\text{H}_2})}{s(s+p_{\text{H}_3})}}{1+KK_{\text{H}}\frac{(s+z_{\text{H}_2})}{s(s+p_{\text{H}_3})}}
\tag{3.5}
$$

Furthermore, we will set $p_{\text{H}_3} = 2C_{\text{d}}v_0$ and obtain

$$
\begin{aligned}
G(s) =& \frac{K(s+z)^2}{s^2(s+p)(s+2C_{\text{d}}v_0)+KK_{\text{H}}s(s+p)(s+z_{\text{H}_2})} \\
&- \frac{KK_{\text{H}}(s+z_{\text{H}_2})}{s(s+2C_{\text{d}}v_0)+KK_{\text{H}}(s+z_{\text{H}_2})}
\end{aligned}
\tag{3.6}
$$

To guarantee stability for the system with a nonlinearity in the sector [0,1], the real part of $G(j\omega)$ has to be greater than -1 for all ω. Thus the limits for

$\omega \to \infty$ and $\omega \to 0$ must be greater than -1 as well. As ω tends to infinity, $\Re(G(j\omega))$ goes to zero, which is of course greater than -1. In turn, for $\omega \to 0$ the limit is

$$\lim_{\omega\to 0} \Re(G(j\omega)) = \frac{-Kz^2(2pC_\mathrm{d}v_0 + KK_\mathrm{H}p + KK_\mathrm{H}z_{\mathrm{H}_2})}{(KK_\mathrm{H}pz_{\mathrm{H}_2})^2} + \frac{2Kz}{KK_\mathrm{H}pz_{\mathrm{H}_2}} - 1 \tag{3.7}$$

Since we require, $\lim_{\omega\to 0} \Re(G(j\omega)) > -1$ it follows that

$$\begin{aligned}
\frac{2Kz}{KK_\mathrm{H}pz_{\mathrm{H}_2}} &> \frac{Kz^2(2pC_\mathrm{d}v_0 + KK_\mathrm{H}p + KK_\mathrm{H}z_{\mathrm{H}_2})}{(KK_\mathrm{H}pz_{\mathrm{H}_2})^2} \\
\frac{2}{z} &> \frac{2pC_\mathrm{d}v_0 + KK_\mathrm{H}p + KK_\mathrm{H}z_{\mathrm{H}_2}}{KK_\mathrm{H}pz_{\mathrm{H}_2}} \\
\frac{2}{z} &> \frac{2C_\mathrm{d}v_0}{KK_\mathrm{H}z_{\mathrm{H}_2}} + \frac{1}{z_{\mathrm{H}_2}} + \frac{1}{p}
\end{aligned} \tag{3.8}$$

As all parameters must be larger than zero, $2C_\mathrm{d}v_0/KK_\mathrm{H}z_{\mathrm{H}_2}$ in (3.8) must be positive, and thus

$$\frac{2}{z} - \frac{1}{p} - \frac{1}{z_{\mathrm{H}_2}} > 0 \quad \Rightarrow \quad z_{\mathrm{H}_2} > \frac{1}{\frac{2}{z} - \frac{1}{p}} = 0.1 \tag{3.9}$$

Selecting $z_{\mathrm{H}_2} = 0.115$ leads to the following limit for K_H

$$\begin{aligned}
\frac{2}{z} - \frac{1}{p} - \frac{1}{z_{\mathrm{H}_2}} &> \frac{2C_\mathrm{d}v_0}{KK_\mathrm{H}z_{\mathrm{H}_2}} \\
KK_\mathrm{H}z_{\mathrm{H}_2} &> \frac{2C_\mathrm{d}v_0}{\frac{2}{z} - \frac{1}{p} - \frac{1}{z_{\mathrm{H}_2}}} \\
K_\mathrm{H} &> \frac{2C_\mathrm{d}v_0}{Kz_{\mathrm{H}_2}\left(\frac{2}{z} - \frac{1}{p} - \frac{1}{z_{\mathrm{H}_2}}\right)} = 0.0023
\end{aligned} \tag{3.10}$$

Picking $K_\mathrm{H} = 0.003$, we find through (3.7) that $\Re(G(j\omega)) \to 0.18$ as $\omega \to 0$. On Figure 3.4 we can verify that $\Re(G(j\omega))$ is greater than -1 for all ω and therefore the circle criterion guarantees stability for the subsystem.

Note that this is only true if we ignore the time delay T_d. Although the Nyquist curve for the system with time delay T_d lies right of the line $\Re = -1$ for all ω as well it is not clear that this guarantees stability for this infinite dimensional problem. However, simulating the system in the following section we will observe stability.

Another possibility for describing and analysing the system as a Lur'e problem is to use state space formulation. Therefore we choose the following state space realisations for $P(s)$, $C(s)$ and $H(s)$

$$\dot{x}_\mathrm{P} = A_\mathrm{P}x_\mathrm{P} + b_\mathrm{P}u_\mathrm{sat} \qquad x = c_\mathrm{P}^\mathrm{T}x_\mathrm{P} \tag{3.11a}$$

$$\dot{x}_\mathrm{C} = A_\mathrm{C}x_\mathrm{C} + b_\mathrm{C}(e - y_\mathrm{H}) \qquad u = c_\mathrm{C}^\mathrm{T}x_\mathrm{C} + d_\mathrm{C}(e - y_\mathrm{H}) \tag{3.11b}$$

$$\dot{x}_\mathrm{H} = A_\mathrm{H}x_\mathrm{H} + b_\mathrm{H}(u - u_\mathrm{sat}) \qquad y_\mathrm{H} = c_\mathrm{H}^\mathrm{T}x_\mathrm{H} \tag{3.11c}$$

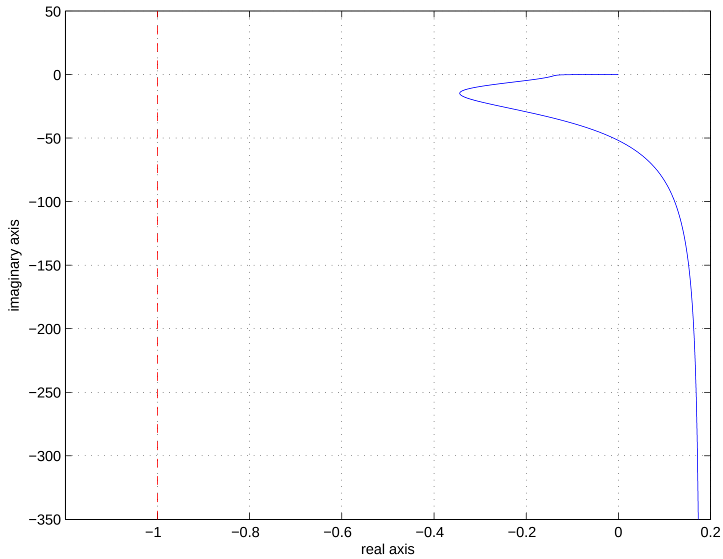


Figure 3.4: Anti-windup system: Nyquist plot of the critical transfer function

With the same realisations for $P(s)$ and $C(s)$ used above and any realisation for $H(s)$

$$\dot{x}_{\mathrm{P}} = \begin{bmatrix} 0 & 1 \\ 0 & -2C_{\mathrm{d}}v_0 \end{bmatrix} x_{\mathrm{P}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_{\mathrm{sat}} \tag{3.12a}$$

$$x = \begin{pmatrix} 1 & 0 \end{pmatrix} x_{\mathrm{P}} \tag{3.12b}$$

$$\dot{x}_{\mathrm{C}} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{T} \end{bmatrix} x_{\mathrm{C}} + \begin{pmatrix} k_{\mathrm{i}} \\ 1 \end{pmatrix} (e - y_{\mathrm{H}}) \tag{3.12c}$$

$$u = \begin{pmatrix} 1 & -\frac{k_{\mathrm{d}}}{T^2} \end{pmatrix} x_{\mathrm{C}} + \left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T} \right) (e - y_{\mathrm{H}}) \tag{3.12d}$$

That leads to

$$\underbrace{\begin{pmatrix} \dot{x}_{\mathrm{P}} \\ \dot{x}_{\mathrm{C}} \\ \dot{x}_{\mathrm{H}} \end{pmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} A_{\mathrm{P}} & 0 & 0 \\ 0 & A_{\mathrm{C}} & -b_{\mathrm{C}} c_{\mathrm{H}}^{\mathrm{T}} \\ 0 & b_{\mathrm{H}} c_{\mathrm{C}}^{\mathrm{T}} & A_{\mathrm{H}} - b_{\mathrm{H}} d_{\mathrm{C}} c_{\mathrm{H}}^{\mathrm{T}} \end{bmatrix}}_{A'} \underbrace{\begin{pmatrix} x_{\mathrm{P}} \\ x_{\mathrm{C}} \\ x_{\mathrm{H}} \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} 0 \\ b_{\mathrm{C}} \\ b_{\mathrm{H}} d_{\mathrm{C}} \end{pmatrix}}_{b} e + \underbrace{\begin{pmatrix} b_{\mathrm{P}} \\ 0 \\ -b_{\mathrm{H}} \end{pmatrix}}_{\hat{b}} u_{\mathrm{sat}} \tag{3.13}$$

Ignoring the reference signal, the error is simply $e = -x = -c_{\mathrm{P}}^{\mathrm{T}} x_{\mathrm{P}} = -c^{\mathrm{T}} X$ and the systems behaviour for $u \neq u_{\mathrm{sat}}$ is described by

$$\dot{X} = \underbrace{(A' - bc^{\mathrm{T}})}_{A_1} X + \hat{b} u_{\mathrm{sat}} \tag{3.14}$$

Note that u can be written as another output of the system

$$\begin{aligned} u &= c_{\mathrm{C}}^{\mathrm{T}} x_{\mathrm{C}} + d_{\mathrm{C}}(e - y_{\mathrm{H}}) \\ &= c_{\mathrm{C}}^{\mathrm{T}} x_{\mathrm{C}} - d_{\mathrm{C}} c_{\mathrm{P}}^{\mathrm{T}} x_{\mathrm{P}} - d_{\mathrm{C}} c_{\mathrm{H}}^{\mathrm{T}} x_{\mathrm{H}} \\ &= -\hat{c}^{\mathrm{T}} X \end{aligned} \tag{3.15}$$

If the actuator signal is not saturated, we obtain the description of the system dynamics by merging (3.14) and (3.15)

$$\dot{X} = \underbrace{(A_1 - \hat{b}\hat{c}^{\mathrm{T}})}_{A_2} X \tag{3.16}$$

Thus, we can specify the problem as a switching system, whose dynamic behaviour is given by A_1 or A_2, respectively. A_2 is Hurwitz, while A_1 has one eigenvalue at the origin and the remaining eigenvalues have negative real parts.

Recently Shorten *et al.* showed in [28] how stability of this singular SISO switching systems can be studied by checking the eigenvalues of $A_1 A_2$. More precisely, for the class of systems considered there exists symmetric, positive definite matrix P satisfying

$$A_1^{\mathrm{T}} P + P A_1 \leq 0 \tag{3.17a}$$

$$A_2^{\mathrm{T}} P + P A_2 < 0 \tag{3.17b}$$

if and only if the matrix product $A_1 A_2$ has no eigenvalues on the negative real axis and exactly one at the origin. In that case the common quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P x$ guarantees stability for any arbitrary switching sequence. The proof of this theorem is given in Appendix B.

Here the eigenvalues of $A_1 A_2$ are $\lambda_1 = 775$, $\lambda_{2,3} = 0.017 \pm 0.048\mathrm{i}$, $\lambda_4 = 0.0017$, $\lambda_5 = \lambda_6 = 0.04$ and $\lambda_7 = 0$. Therefore there exists a common quadratic Lyapunov function and the system is stable.

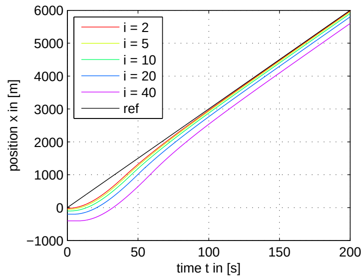
3.2.3 Simulation Results

Again the two manoeuvres introduced earlier were used to test the behaviour of the string using the discussed anti-windup scheme in every subsystem.

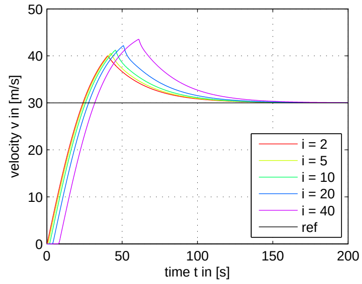
The actuator signal was limited according to (3.1) on page 27 with $u_{\min} = -8\,\mathrm{m/s^2}$ and $u_{\max} = 1.5\,\mathrm{m/s^2}$. Since the cars cannot accelerate arbitrarily rapidly due to the saturated actuator signal, they need a fairly long time to reach their desired positions and drive with maximum acceleration for $\approx 50\,\mathrm{s}$. While the error signals do not appear to grow with the string length in the first manoeuvre, they clearly grow in the second manoeuvre. Because the disturbance is fairly small, the actuator signal for the first 30 subsystems is not saturated and the string therefore behaves like a linear system. Thus, string instability is unavoidable for small signals.

3.2.4 Discussion of String Stability

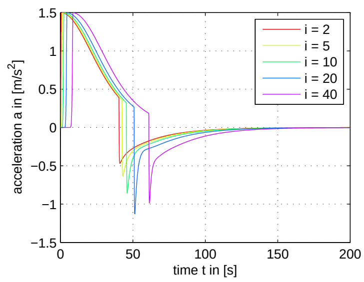
To analyse string stability we use the state space realisation in (3.13) with $e_i = x_{i-1} - x_i - x_{\mathrm{d}} = c^{\mathrm{T}}(X_{i-1} - X_i) - x_{\mathrm{d}}$. The error states are $\xi_i = X_{i-1} - X_i - c x_{\mathrm{d}}$ for



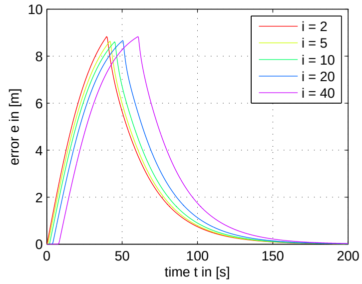
(a) Position x



(b) Velocity v

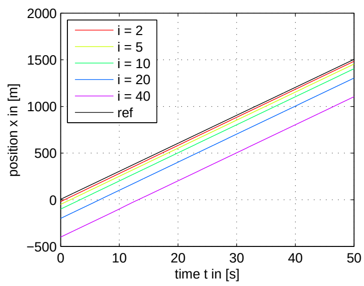


(c) Acceleration a

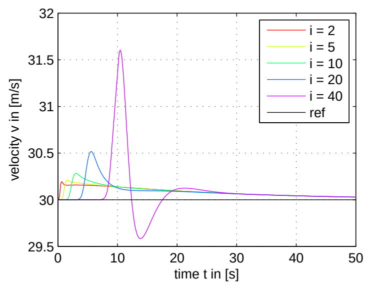


(d) Error e

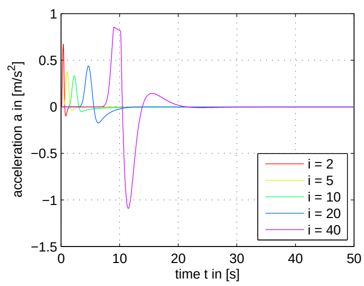
Figure 3.5: Anti-windup system: Simulation of manoeuvre I



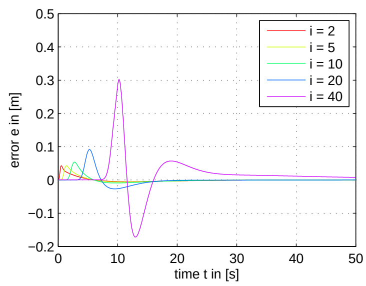
(a) Position x



(b) Velocity v



(c) Acceleration a



(d) Error e

Figure 3.6: Anti-windup system: Simulation of manoeuvre II

all $i > 1$ and $\xi_1 = X_\mathrm{r} - X_1 - cx_\mathrm{d}$ with $X_\mathrm{r} = \begin{pmatrix} x_\mathrm{r} & v_\mathrm{r} & 2C_\mathrm{d}v_0v_\mathrm{r} & 0 & 0 & 0 & 0 \end{pmatrix}^\mathrm{T}$. Differentiating ξ_1 and ξ_i results in

$$\begin{aligned}
\dot{\xi}_1 &= \dot{X}_\mathrm{r} - A'X_1 - \hat{b}u_{\mathrm{sat}_1} - bc^\mathrm{T}(X_\mathrm{r} - X_1) + bx_\mathrm{d} \\
&= A'\xi_1 - AX_\mathrm{r} + \dot{X}_\mathrm{r} + A'cx_\mathrm{d} - \hat{b}u_{\mathrm{sat}_1} - bc^\mathrm{T}\xi_1 - bc^\mathrm{T}cx_\mathrm{d} + bx_\mathrm{d} \qquad (3.18a) \\
\dot{\xi}_i &= A'X_{i-1} - A'X_1 + \hat{b}u_{\mathrm{sat}_{i-1}} - \hat{b}u_{\mathrm{sat}_i} \\
&\qquad + bc^\mathrm{T}(X_{i-2} - X_{i-1}) - bc^\mathrm{T}(X_{i-1} - X_i) - bx_\mathrm{d} + bx_\mathrm{d} \\
&= A'\xi_i + A'cx_\mathrm{d} + bc^\mathrm{T}\xi_{i-1} - bc^\mathrm{T}\xi_i + \hat{b}u_{\mathrm{sat}_{i-1}} - \hat{b}u_{\mathrm{sat}_i} \qquad (3.18b)
\end{aligned}$$

Note again that $c^\mathrm{T}c = 1$ and $A'c = 0$. Moreover, choosing the state space realisation given in (3.12) and (3.13) with the reference vector X_r specified above, $\dot{X}_\mathrm{r} - AX_\mathrm{r} = \hat{b}(2C_\mathrm{d}v_0v_\mathrm{r})$ and (3.18) can be simplified to

$$\dot{\xi}_1 = \underbrace{\left(A' - bc^\mathrm{T}\right)}_{A}\xi_1 - \hat{b}(u_{\mathrm{sat}_1} - 2C_\mathrm{d}v_0v_\mathrm{r}) \qquad (3.19)$$

$$\dot{\xi}_i = \underbrace{\left(A' - bc^\mathrm{T}\right)}_{A}\xi_i + bc^\mathrm{T}\xi_{i-1} - \hat{b}(u_{\mathrm{sat}_i} - u_{\mathrm{sat}_{i-1}}) \qquad (3.20)$$

Although the initial conditions of ξ_i are zero for all $i > 1$ for both manoeuvres, the system cannot be transformed to an initial condition problem, due to the driving term $\tilde{b}$:

$$\underbrace{\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_N \end{pmatrix}}_{\dot{\xi}} = \underbrace{\begin{bmatrix} A & & & 0 \\ bc^\mathrm{T} & A & & \\ & \ddots & \ddots & \\ 0 & & bc^\mathrm{T} & A \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix}}_{\xi} - \underbrace{\begin{pmatrix} \hat{b}(u_{\mathrm{sat}_1} - 2C_\mathrm{d}v_0v_\mathrm{r}) \\ \hat{b}(u_{\mathrm{sat}_2} - u_{\mathrm{sat}_1}) \\ \vdots \\ \hat{b}(u_{\mathrm{sat}_N} - u_{\mathrm{sat}_{N-1}}) \end{pmatrix}}_{\tilde{b}} \qquad (3.21)$$

Note that $2C_\mathrm{d}v_0v_\mathrm{r} = 1.25\ \mathrm{m/s^2}$ is smaller than $u_\mathrm{max} = 1.5\ \mathrm{m/s^2}$ and all actuator signals are limited by the same maximal and minimal values u_max and u_min. Hence all sector nonlinearities in $\tilde{b}$ still lie in sector [0,1].

The higher dimensional circle criterion could be used to analyse the stability of the system given in (3.21). Nevertheless we do not expect string stability according to Definition 1 or Definition 2 because these definitions require a form of uniform stability – independent of the string length N. Since the circle criterion guarantees stability for all possible nonlinearities within the sector (which is here [0,1]), it would also assure stability for the linear case studied in Chapter 1. As we have already seen that a simple linear unit feedback controller in this case can never provide string stability, it is not possible to guarantee string stability using the circle criterion.

Furthermore, in Figure 3.6 we saw that error signals grow with the position within the string for small disturbances. Thus, the system satisfies neither Definition 1 nor Definition 2.

If the error signals saturate beyond a certain string length we could only hope for "string boundedness". That means that small disturbances may still grow while propagating through the string. However, the largest peak of the deviation states of a subsystem does not increase after a certain string length.

3.3 Signed Q Term

In the previous section we have seen how saturation of the actuator signal and an anti-windup scheme helped to improve string behaviour. However, the problem of string stability was not solved with this controller approach.

Yanakiev *et al.* used in [36] a PIDQ-controller (that is a PID-controller with an additional signed quadratic term Q) for speed tracking (forcing the velocity error to zero) and an adaptive PIQ-controller for vehicle following (forcing the position error to zero) with a fairly detailed truck model. Here we use a simple PIDQ-controller with fixed gains.

A simple controller with a signed quadratic term can be described by one of the two schemes shown in Figure 3.7 where $C_1(s)$ and $C_2(s)$ are proper, linear transfer functions and $Q(\mu)$ is the signed quadratic term of the form

$$Q(\mu) = k_{\mathrm{q}}|\mu|\mu \tag{3.22}$$

Note that $Q(\mu)$ can be expressed as a sector nonlienarity in the sector $[0,\infty)$.

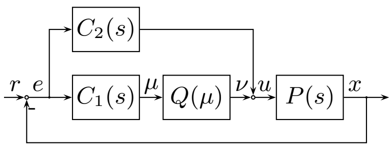


Figure 3.7: PIDQ-controller: Block diagrams of both possible realisations

To guarantee stability for the subsystem the Nyquist curve of the critical transfer function from ν to μ $G(s)$ must not have negative real parts.

Although the two schemes displayed in Figure 3.7 are different the critical transfer function $G(s)$ from ν to μ is identical (when the reference signal is ignored)

$$\begin{aligned} G(s) &= \frac{P(s)C_1(s)}{1+P(s)C_2(s)} \\ &= \frac{\frac{1}{s(s+2C_{\mathrm{d}}v_0)}C_1(s)}{1+\frac{1}{s(s+2C_{\mathrm{d}}v_0)}C_2(s)} \\ &= \frac{\frac{1}{s(s+2C_{\mathrm{d}}v_0)}\frac{z_1(s)}{p_1(s)}}{1+\frac{1}{s(s+2C_{\mathrm{d}}v_0)}\frac{z_2(s)}{p_2(s)}} \\ &= \frac{z_1(s)p_2(s)}{p_1(s)\big[s(s+2C_{\mathrm{d}}v_0)p_2(s)+z_2(s)\big]} \end{aligned} \tag{3.23}$$

For any proper transfer function for $C_1(s)$ and $C_2(s)$ the order of $z_1(s)$ or $z_2(s)$ cannot be greater than the number of poles in $p_1(s)$ or $p_2(s)$, respectively. Hence, for any proper, minimum phase stable approach for $C_1(s)$ and $C_2(s)$, $G(s)$ will cross the imaginary axis at least one time. Thus the circle criterion cannot be used to guarantee stability for a sector nonlinearity in sector $[0,\infty)$.

However, for some cases the Popov criterion, see for instance [12, 34], may be used. First we need to transform $G(j\omega)$ into

$$G'(j\omega) = \Im(G(j\omega)) + j\omega\Re(G(j\omega)) \tag{3.24}$$

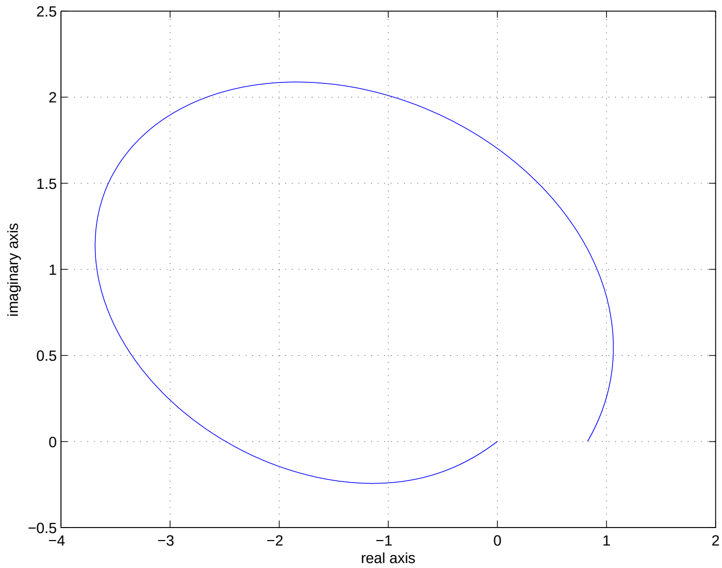


Figure 3.8: PIDQ-controller: Popov plot of the critical transfer function

If we find a $\eta \geq 0$ such that the Nyquist curve of $G'(j\omega)$ lies to the right of the line that crosses the real axis at $-\frac{1}{k}$ with slope $\frac{1}{\eta}$, the system described in Figure 3.2 will be stable for any time invariant nonlinearity $\phi(y)$ within the sector $[0,k]$.

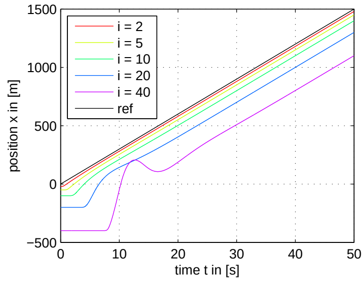
Since we would like to recover the original PID-controller for $k_q \to 0$ we must choose $C_2(s)$ to be $C(s)$ from (1.12) on page 8. We have investigated different choices for $C_1(s)$. Unfortunately, we have not been able to find an approach that satisfies the Popov criterion for the sector $[0,\infty)$ and gives reasonable good simulation results.

For instance, when selecting

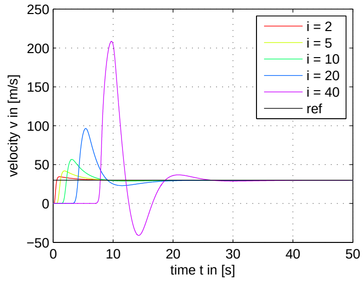
$$C_1(s) = K\frac{s+z}{s+p} \qquad \text{and} \qquad C_2(s) = K\frac{(s+z)^2}{s(s+p)} \tag{3.25}$$

where z, p and K are the parameters given by the linear controller in (1.12) the Popov criterion does not guarantee stability for the sector $[0,\infty)$ (see Figure 3.8). Stability might only be guaranteed locally for a subset of all possible initial conditions or for nonlinearities in a smaller sector $[0,k]$.

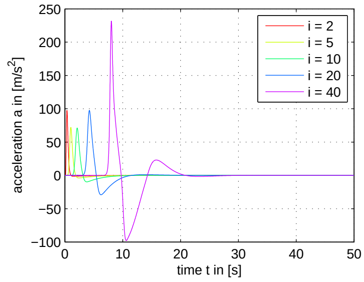
Nevertheless, we observed stability simulating both manoeuvres introduced earlier setting $k_q = 0.002$. Simulations of the first manoeuvre (see Figure 3.9) showed slightly better behaviour than simulations of the linear approach. However, in the second manoeuvre (Figure 3.10) using the PIDQ-controller gave even worse results than for the linear controller. Therefore, we shall not pursue this approach any further.



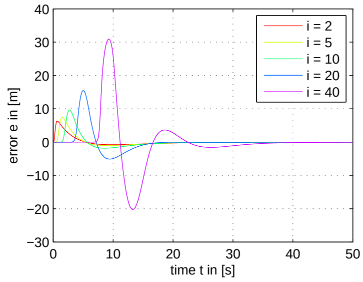
(a) Position x



(b) Velocity v

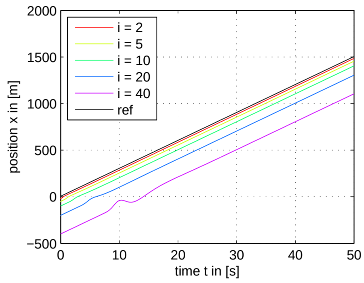


(c) Acceleration a

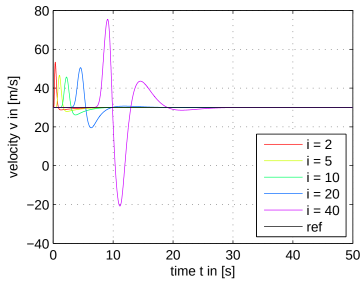


(d) Error e

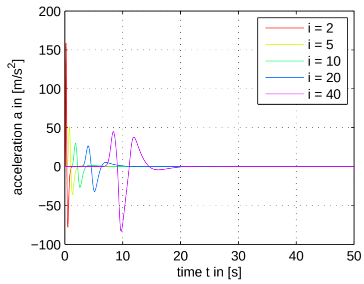
Figure 3.9: PIDQ-controller: Second approach: Simulation of manoeuvre I



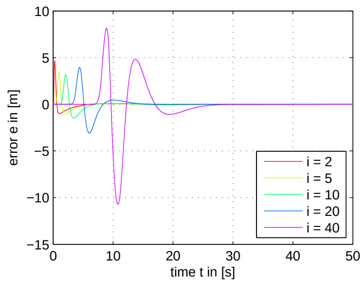
(a) Position x



(b) Velocity v



(c) Acceleration a



(d) Error e

Figure 3.10: PIDQ-controller: Second approach: Simulation of manoeuvre II

Chapter 4

Variable Time Headway

Since the nonlinear controllers introduced so far could not solve the string instability problem, we will focus again on using a time headway. Since we have observed undesirable large steady state separation and slow dynamics we will now show how to improve these issues with a variable time headway.

4.1 Introduction

In Chapter 2 we showed how to guarantee L_∞- and L_2-string stability using a constant time headway h. However, several undesirable effects such as large separations between the cars in steady state and comparatively slow system dynamics cannot be avoided using a fixed time headway if string stability is to be guaranteed. Yanakiev*et al.* therefore introduced a variable time headway, [36].

In this chapter we will use this approach and replace the fixed time headway h by h_{var} which depends on the relative velocity between the vehicle and its predecessor.

We will show stability of the subsystem using the circle criterion.

Simulating the string we will observe that a smaller variable time headway than a fixed time headway is necessary to obtain string stability and we shall discuss string stability for this approach.

4.2 Controller Design with Variable Time Headway

When a vehicle is moving faster than the car in front, a larger time headway is needed to prevent crashes. Otherwise, if the vehicle is driving slower than its predecessor, a smaller time headway is sufficient to obtain string stability and will speed up system dynamics. To ensure positive distances between the vehicles for small relative velocities, a fixed part is added to the variable time headway.

$$h_{\mathrm{var}}(v,v_{\mathrm{l}}) = \begin{cases} 1 & \text{for } v \geq v_{\mathrm{l}} + \frac{1-h_0}{k_{\mathrm{h}}} \\ h_0 + k_{\mathrm{h}}(v - v_{\mathrm{l}}) & \text{for } v_{\mathrm{l}} - \frac{h_0}{k_{\mathrm{h}}} < v < v_{\mathrm{l}} + \frac{1-h_0}{k_{\mathrm{h}}} \\ 0 & \text{for } v \leq v_{\mathrm{l}} - \frac{h_0}{k_{\mathrm{h}}} \end{cases} \tag{4.1}$$

Note that v_l is the velocity of the preceding vehicle, which can be the reference velocity v_r for the first or v_{i-1} for the following cars.

The product of the variable time headway h_var (v,v_l) and the velocity v can be described as a nonlinear function $\phi(v,v_\mathrm{l})$

$$\phi(v,v_\mathrm{l}) = \begin{cases} v & \text{for } v \geq v_\mathrm{l} + \frac{1-h_0}{k_\mathrm{h}} \\ \big(h_0 + k_\mathrm{h}(v - v_\mathrm{l})\big)v & \text{for } v_\mathrm{l} - \frac{h_0}{k_\mathrm{h}} < v < v_\mathrm{l} + \frac{1-h_0}{k_\mathrm{h}} \\ 0 & \text{for } v \leq v_\mathrm{l} - \frac{h_0}{k_\mathrm{h}} \end{cases} \tag{4.2}$$

As in Chapter 2 we introduce the additional transfer function $Q(s) = \frac{1}{h_0 s+1}$, that yields to the system displayed in Figure 4.1.

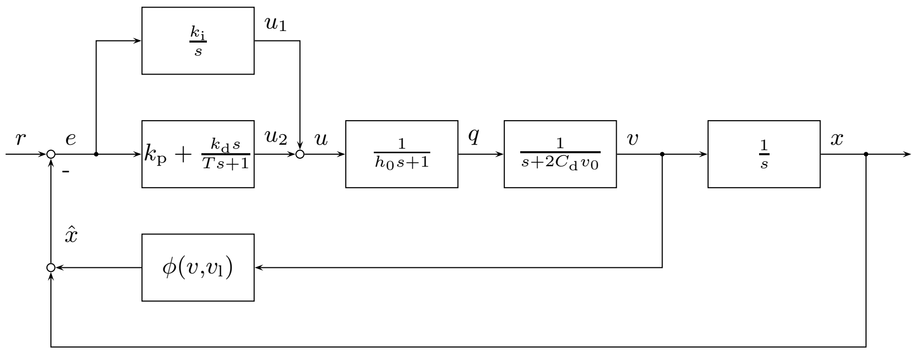


Figure 4.1: Variable time headway: Block diagram

4.3 Stability of Subsystem

To analyse the stability of the subsystem, the function $\phi(v,v_\mathrm{l})$ is described as a sector nonlinearity. Note that Figure 4.2 shows ϕ for $v_\mathrm{l} = h_0/k_\mathrm{h}$. Although $\phi(v,v_\mathrm{l})$ will defer with different values of v_l the function $\phi(v,v_\mathrm{l})$ always lies in sector [0,1] since $0 \leq h_\mathrm{var}(v,v_\mathrm{l}) \leq 1$.

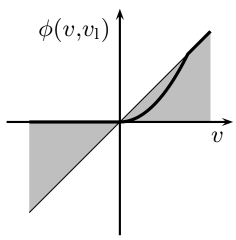


Figure 4.2: Lur'e problem: Variable time headway as a sector nonlinearity

As in Section 3.2 we will check stability of the subsystem using the circle criterion. Note that the Popov criterion cannot be applied in this case, because the nonlinearity ϕ depends on v_l and therefore is time varying.

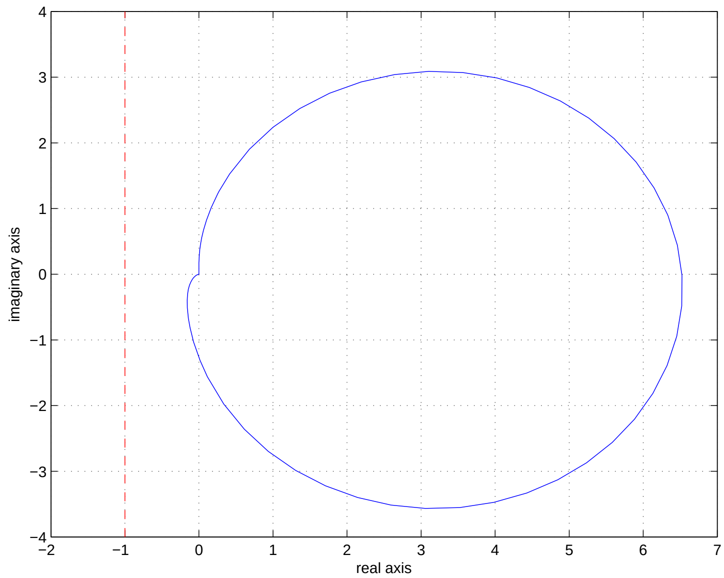


Figure 4.3: Variable time headway: Nyquist plot of the critical transfer function

To guarantee absolut stability of the subsystem, the Nyquist plot of the critical transfer function $G(j\omega)$ must lie to the right of the vertical line $\Re = -1$. Or, in other words, the real part $\Re(G(j\omega))$ must be greater than -1 for all ω.

Since we will use the same PID-controller as above and the same model for the plant, the critical transfer function is always

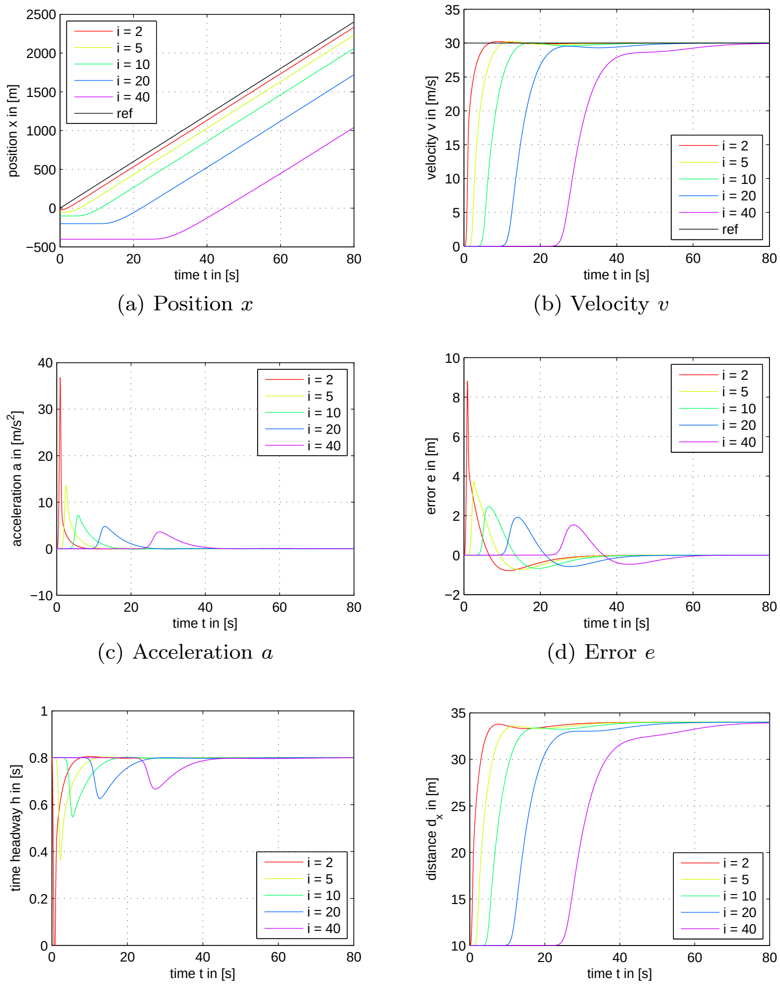
$$G(s) = \frac{C(s)Q(s)\frac{1}{s+2C_\mathrm{d}v_0}}{1 + C(s)Q(s)P(s)} \tag{4.3}$$

In fact, the real part $\Re(G(j\omega))$ is greater than -1 for all ω (see Figure 4.3) and thus the system satisfies the circle criterion for any nonlinearity in sector [0,1]. Note again that this result may not hold for a system with time delay. However, in the following simulation results we will observe stability.

4.4 Simulation Results

The following two figures show the simulation of manoeuvre I and manoeuvre II discussed above using a variable time headway with $h_0 = 0.8$ s and $k_\mathrm{h} = 0.05\ \mathrm{s}^2/\mathrm{m}$.

In the first manoeuvre we can observe that beyond certain string length cars do not drive faster than $v_\mathrm{r} = 30$ m/s. Thus those vehicles do not brake what is more energy efficient. Besides a much smaller distance between the vehicles ($d_\mathrm{x} = 34$ m in steady state) is maintained compared to the steady state distance of 77.1 m when using a constant time headway of h_∞ (see Figure 2.7 on page 24).



(a) Position x (b) Velocity v

(c) Acceleration a (d) Error e

(e) Time Headway h (f) Distance d_x

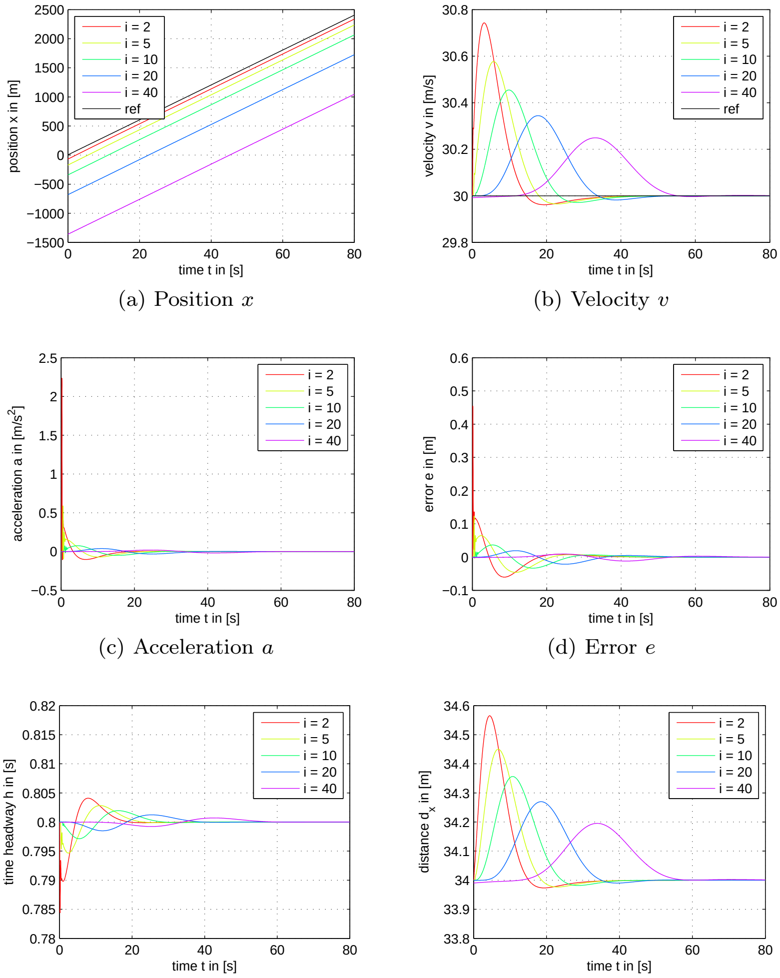
Figure 4.4: Variable time headway: Simulation of manoeuvre I

Since the distance between two vehicles is never smaller than the fixed part of $x_\mathrm{d} = 10$ m and is not decreasing with the string length, x_d may be reduced to improve traffic throughput (see Subfigure 4.4f and Subfigure 4.5f).

4.5 Discussion of String Stability

Although h_0 is much smaller than h_2 and h_∞, it seems to ensure L_∞-string stability according to Definition 1.

To discus string stability the system has to be transformed into error coordinates. Choosing the same state vector X_i as in (2.4) on page 16 would



(a) Position x (b) Velocity v

(c) Acceleration a (d) Error e

(e) Time Headway h (f) Distance d_x

Figure 4.5: Variable time headway: Simulation of manoeuvre II

provoke a problem differentiating its first state e_i

$$e_i = x_{i-1} - x_i - x_\mathrm{d} - \phi(v_i,v_{i-1}) \tag{4.4}$$

$$\dot{e}_i = v_{i-1} - v_i - \dot{\phi}(v_i,v_{i-1}) \tag{4.5}$$

since $\dot{\phi}(v_i,v_{i-1})$ is nonlinear and we will have to distinguish between the three different cases in (4.1) on page 39.

To avoid the derivative of ϕ we can choose the position x_i as the first entry of X_i. However, it would not be possible to find an error state vector ξ_i to guarantee $\xi_i(0) = 0$ for all $i > 1$ and $\xi_i \to 0$ as t goes to infinity for all i for the first manoeuvre. For the second manoeuvre Definition 1 and Definition 2 could be used to analyse the system with the error state vector $\xi_i = X_{i-1} - X_i - c(x_\mathrm{d} + h_0 v_\mathrm{r})$ with c being the first unit vector.

Even though with a suitable coordinate transformation the system would have a form similar to (3.21) on page 34. Using the higher dimensional circle criterion would not give any useful result since the sector [0,1] includes the case with zero time headway also. Hence, to find an analytic solution for the minimal h_0 needed to guarantee string stability other linear approaches which take the particular form of the nonlinearity into account are needed.

Conclusions and Future Directions

In the first chapter we have presented a definition of L_2- and L_∞-string stability. We have introduced a simple vehicle model and created a PID-controller to guarantee stability of the subsystem composed of the vehicle model and the controller. As expected we have observed string instability, in that in both simulated manoeuvres the disturbances grow while travelling along the string. We have also shown that this conclusion is independent of the particular manoeuvre or the linear controller used.

In Chapter 2 we therefore have introduced a time headway. Instead of requiring a fixed inter-vehicle distance, a combination of fixed and velocity dependent distance has been proposed. We showed that using a sufficiently large time headway guarantees that every state in the system satisfies the requirements given in the definition of string stability. We have also derived sufficient conditions on the time headways needed to guarantee L_2- or L_∞-string stability, respectively. This was verified in simulations but undesired effects such as large distances between the vehicles in steady state and slow system dynamics became apparent.

Two nonlinear controllers have been analysed and simulated in Chapter 3. First we have used an anti-windup scheme and limited the actuator signal to within realistic ranges. Although we could prove stability for the subsystem without time delay using the circle criterion, the stability result may not hold for the complete system with time delay. However, all subsystems in the simulation of both manoeuvres were stable. Since the disturbances in the second manoeuvre have grown while propagating along the string neither L_2- nor L_∞-string stability appeared unlikely.

We also introduced a controller with a signed quadratic term. Stability of a simplified subsystem could be shown using the Popov criterion. However, the system behaviour observed in both simulations was very poor and therefore gave no reason to believe that string stability might be possible.

Finally, in Chapter 4 we have modified the approach taken in Chapter 2. Instead of a fixed time headway a combination of a fixed and a variable time headway which depends on the relative velocity between every vehicle and its predecessor was introduced. Stability of a simplified subsystem consisting of vehicle model ignoring the time delay, a PID-controller and the variable time headway in the feedback path could be proved with the circle criterion. Even though simulations have shown string stability in both manoeuvres (with a smaller fixed time headway than the headways used in Chapter 2) proving string stability with the methods used in this work was not possible.

As for future directions, it would be interesting to extend our results to the definition of string stability given by Swaroop, where bounded signals are required for every bounded vector of initial conditions instead of considering zero initial conditions for all but the first subsystem.

Since solving the string stability problem with the nonlinear controllers discussed above was not possible other nonlinear approaches may allow such analysis.

Until now detailed information about the nonlinear vehicle model and the time delay have been mostly ignored. Using more accurate and possibly nonlinear models may also allow a more extensive study of the system.

However, in order to obtain more precise results other methods may be needed that take into account the particular form of the nonlinearities (in the plant and the controller) instead of just assuming sector bounded nonlinearities. Using information about the particular form of the variable time headway may lead to an explicit formula for the minimum fixed time headway that guarantees string stability.

Although the nonlinear controllers proposed in this work could not ensure string stability for the particular vehicle model used, more promising results may be achieved using a different plant model. This might also be the case when looking at models from completely different areas of application, such as irrigation flow systems or supply chains.

Appendix A

Vector, Matrix and Operator Norms

The following definitions of vector, matrix and operator norms are widely known and can be found in most textbooks, see e.g. [9, 23, 33, 34, 37]. However they are displayed here to avoid misunderstandings and to clarify the notation used in this thesis.

A.1 Vector Norms

Let $x \in \mathbb{C}^n$ be a vector of n complex numbers. The L_p-norms are defined as

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \qquad \forall 1 \le p < \infty \tag{A.1a}$$

$$||x||_\infty = \max_i |x_i| \tag{A.1b}$$

Note that the following inequalities hold

$$||x||_2 \le ||x||_1 \le \sqrt{n}||x||_2 \tag{A.2a}$$

$$||x||_\infty \le ||x||_2 \le \sqrt{n}||x||_\infty \tag{A.2b}$$

$$||x||_\infty \le ||x||_1 \le n||x||_\infty \tag{A.2c}$$

Let $x(\,\cdot\,) : T \to \mathbb{C}^n$ be a complex vector valued function of time over the interval $T \subset \mathbb{R}$. The L_p-norms are defined as

$$||x(\,\cdot\,)||_{L_p} = \left(\int_T \sum_{i=1}^{n} |x_i(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} \qquad \forall 1 \le p < \infty \tag{A.3a}$$

$$||x(\,\cdot\,)||_{L_\infty} = \sup_i \operatorname*{ess\,sup}_{t \in T} |x_i(t)| \tag{A.3b}$$

The same definitions for scalar signal L_p-norms hold for $n = 1$. Note that we will distinguish between $||x(\,\cdot\,)||_{L_p}$ defined in (A.3) and $||x(t)||_p$, which is the simple vector norm defined in (A.1) at a certain time t.

A.2 Matrix Norms

Let $A^{m\times n} \in \mathbb{C}^{m\times n}$ be a complex, time invariant matrix. Then the *induced* L_p-norms are defined as

$$||A||_p = ||A||_{\mathrm{i}_p} = \sup_{x\neq 0} \frac{||Ax||_p}{||x||_p} \tag{A.4}$$

In particular

$$||A||_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| \tag{A.5a}$$

$$||A||_2 = \sqrt{\lambda_{max}(A^*A)} \tag{A.5b}$$

$$||A||_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}| \tag{A.5c}$$

Note that the following inequalities hold

$$\frac{1}{\sqrt{m}}||A||_1 \leq ||A||_2 \leq \sqrt{n}||A||_1 \tag{A.6a}$$

$$\frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq \sqrt{m}||A||_\infty \tag{A.6b}$$

$$\frac{1}{m}||A||_1 \leq ||A||_\infty \leq n||A||_1 \tag{A.6c}$$

$$||A||_2 \leq \sqrt{||A||_1||A||_\infty} \tag{A.6d}$$

A.3 Operator Norms

Let $G(s)$ be a stable, proper, finite dimensional, linear, time invariant operator and $g(t)$ the corresponding impulse response. Then the *induced* L_p-norms are defined as

$$||G||_{\mathrm{i}_p} = \sup_{x\neq 0} \frac{||Gx||_{L_p}}{||x||_{L_p}} \tag{A.7}$$

In particular for $p = 2$

$$||G||_{\mathrm{i}_2} = \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \sigma_{\max}[G(j\omega)] = ||G||_\infty = ||G||_{H_\infty} \tag{A.8}$$

For scalar operators the induced ∞- and 2-norm can be simplified

$$||G||_{\mathrm{i}_2} = \sup_{\omega\in\mathbb{R}} |G(j\omega)| \tag{A.9a}$$

$$||G||_{\mathrm{i}_\infty} = \int_0^\infty |g(t)|\mathrm{d}t = ||g(t)||_{L_1} \tag{A.9b}$$

Unfortunately the induced ∞-norm for a scalar operator, is the same as the L_1-norm over its impulse response. Moreover, the induced 2-norm is the supremum of the absolute value of $G(j\omega)$ and therefore is often called H_∞- or ∞-norm.

If an operator $G(j\omega)$ is applied on a fixed vector x rather than on a vector valued function $x(\cdot)$, the induced norm is defined as

$$
\begin{aligned}
||G||_{H_2} &= ||G||_2 \\
&= \sup_{x \neq 0} \frac{||Gx||_{L_2}}{||x||_2} \\
&= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Trace}[G^*(j\omega)G(j\omega)]\mathrm{d}\omega \right)^{\frac{1}{2}} \qquad \text{(A.10)}
\end{aligned}
$$

Appendix B

Quadratic stability and singular SISO switching systems

Since the article *Quadratic stability and singular SISO switching systems* by R. Shorten, M. Corless, K. Wulff, S. Klinge and R. Middleton has not been published yet, we shall reproduce its mainpart here. The work was submitted to the *IEEE Transactions on Automatic Control* in August 2008, [28].

Consider a switching system described by

$$\dot{x} = \left[A - \sigma(t,x)gh^T\right]x \tag{B.1}$$

where the state $x(t)$ and g, h are real vectors, A is a real square matrix, and the scalar switching function σ satisfies

$$0 \leq \sigma(t,x) \leq 1\,. \tag{B.2}$$

Suppose A is a Hurwitz matrix, that is, all its eigenvalues have negative real parts and that all the eigenvalues of $A - gh^T$ have negative real parts except for a single eigenvalue at zero. We can guarantee that the switching system (B.1) is stable about the origin and all solutions are bounded if there is a real symmetric positive definite matrix P satisfying the following two Lyapunov matrix inequalities.

$$A^T P + PA < 0 \tag{B.3a}$$

$$(A - gh^T)^T P + P(A - gh^T) \leq 0\,. \tag{B.3b}$$

In this short note, we show that the following simple condition is both necessary and sufficient condition for the existence of a common Lyapunov matrix P.

The matrix product $A(A - gh^T)$ has no eigenvalues at the negative real axis and only one zero eigenvalue.

B.1 Strictly positive real transfer functions

Before obtaining our main result, we obtain some preliminary results on strictly positive real (SPR) transfer functions. In everything that follows, A is a real $n \times n$ matrix and b,c are real n-vectors.

Recall that a scalar transfer function H is strictly positive real (SPR) if there exists a scalar $\alpha > 0$ such that H is analytic in the region of the complex plane for which $Re(s) \geq -\alpha$ and

$$H(j\omega - \alpha) + H(j\omega - \alpha)^* \geq 0 \qquad \text{for all} \qquad \omega \in \mathbb{R}\,. \tag{B.4}$$

We say H is regular if $H(j\omega) + H(j\omega)^*$ is not identically zero for all $\omega \in \mathbb{R}$. For convenience, we will include regularity as a requirement for SPR.

The following standard result provides a more convenient characterization of SPR. It eliminates α.

Lemma 1. *[12] Suppose A is Hurwitz. Then the transfer function $H(s) = c^T(sI - A)^{-1}b$ is SPR if and only if*

$$H(j\omega) + H(j\omega)^* > 0 \qquad \textit{for all } \ \omega \in \mathbb{R} \tag{B.5}$$

$$\lim_{\omega \to \infty} \omega^2 \big[H(j\omega) + H(j\omega)^*\big] > 0. \tag{B.6}$$

In checking SPR of a system it is sometimes more convenient to check SPR of a system which is equivalent (from an SPR viewpoint) to the original system. The following lemma provides such an equivalent system and is useful for generating some of the results of this paper. This system is simply obtained by replacing A with A^{-1}.

Lemma 2. *The transfer function $H(s) = c^T(sI - A)^{-1}b$ is SPR if and only if the $H_I(s) = c^T(sI - A^{-1})^{-1}b$ is SPR.*

Proof : Suppose H is SPR. The identity $(sI - A^{-1})^{-1} = s^{-1}I - s^{-2}(s^{-1}I - A)^{-1}$ implies that

$$H_I(s) = s^{-1}c^T b - s^{-2}H(s^{-1})\,; \tag{B.7}$$

hence,

$$H_I(j\omega) + H_I(j\omega)^* = \omega^{-2}[H(-j\omega^{-1}) + H(-j\omega^{-1})^*] > 0 \qquad \text{for all} \quad \omega \neq 0\,.$$

Considering limits as $\omega \to 0$,

$$H_I(0) + H_I(0)^* = \lim_{\tilde{\omega} \to \infty} \tilde{\omega}^2 [H(j\tilde{\omega}) + H(j\tilde{\omega})^*] > 0$$

Finally, we note that

$$\lim_{\omega \to \infty} \omega^2 [H_I(j\omega) + H_I(j\omega)^*] = H(0) + H(0)^* > 0\,.$$

□

The core of our main result is based on a spectral condition for strict positive realness [30]. This result follows as an immediate consequence of the following lemma.

Lemma 3. *[2, 21, 29] Let $H(s) = d + c^T(sI - A)^{-1}b$ where A is invertible. Then, $H(s^{-1}) = \bar{d} + \bar{c}^T(sI - \bar{A})^{-1}\bar{b}$ with $\bar{A} = A^{-1}$, $\bar{b} = -A^{-1}b$, $\bar{c}^T = c^T A^{-1}$ and $\bar{d} = d - c^T A^{-1}b$.*

Proof : Using the definitions in the lemma,

$$\begin{aligned}\bar{d} + \bar{c}^T(sI - \bar{A})^{-1}\bar{b} &= d - c^T A^{-1} b - c^T A^{-1}(sI - A^{-1})^{-1} A^{-1} b \\ &= d - c^T A^{-1}\left(I + (sI - A^{-1})^{-1} A^{-1}\right) b \\ &= d - c^T A^{-1}(sA - I)^{-1} sAb \\ &= d + c^T(s^{-1}I - A)^{-1} b,\end{aligned}$$

which proves the assertion of the lemma.

□

Comment : Note that when H is SPR we must have $\bar{d} > 0$. This follows from the fact that $\bar{d} = H(0)$ and $H(0) + H(0)^* > 0$ since H is SPR.

Now we give the aforementioned spectral characterisation of strict positive realness.

Theorem 1. *Suppose A is Hurwitz. Then, the following statements are equivalent.*

(a) The transfer function $H(s) = c^T(sI - A)^{-1}b$ is SPR.

(b) $c^T A^{-1} b < 0$ and the matrix product $A^{-1}\left(A^{-1} - \frac{A^{-1}bc^TA^{-1}}{c^TA^{-1}b}\right)$ has no negative real eigenvalues and exactly one zero eigenvalue.

(c) $c^T Ab < 0$ and the matrix product $A\left(A - \frac{Abc^TA}{c^TAb}\right)$ has no negative real eigenvalues and exactly one zero eigenvalue.

Proof : In what follows it is convenient to work with $H(s^{-1})$ as in Lemma 3. In particular, the conditions for SPR of H may be restated in terms of the transfer function $H(s^{-1})$. Specifically, conditions (B.5) and (B.6) for SPR are equivalent to

$$\lim_{\omega\to\infty} H(-j\omega^{-1}) + H(-j\omega^{-1})^* > 0 \tag{B.8}$$

$$H(-j\omega^{-1}) + H(-j\omega^{-1})^* > 0 \qquad \forall\, \omega \in \mathbb{R} \quad \omega \neq 0 \tag{B.9}$$

$$\lim_{\omega\to 0} \frac{1}{\omega^2}\left[H(-j\omega^{-1}) + H(-j\omega^{-1})^*\right] > 0 \tag{B.10}$$

Condition (B.8) is equivalent to $c^T A^{-1} b < 0$.

Now consider conditions (B.9) and (B.10). Since A is invertible, Lemma 3 tells us that

$$H(-j\omega^{-1}) = \bar{d} + \bar{c}^T(j\omega I - \bar{A})^{-1}\bar{b} \tag{B.11}$$

with $\bar{A}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ defined in Lemma 3. Using the results in [11] we have

$$\bar{c}^T(j\omega I - \bar{A})^{-1}\bar{b} + \left[\bar{c}^T(j\omega I - \bar{A})^{-1}\bar{b}\right]^* = -2\bar{c}^T(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{b}$$

Since $\bar{d} = -c^T A^{-1} b > 0$, we can write

$$\begin{aligned}H(-j\omega^{-1}) + H(-j\omega^{-1})^* &= 2\bar{d}\det\left[1 - \frac{1}{\bar{d}}\bar{c}^T(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{b}\right] \\ &= 2\bar{d}\det\left[I - \frac{1}{\bar{d}}(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{b}\bar{c}^T\right] \\ &= 2\bar{d}\det\left[(\omega^2 I + \bar{A}^2)^{-1}\right]\,\det\left[\omega^2 I + \bar{A}^2 - \frac{1}{\bar{d}}\bar{A}\bar{b}\bar{c}^T\right]\end{aligned}$$

Thus,

$$H(-j\omega^{-1}) + H(-j\omega^{-1})^* = \frac{2\bar{d}\det[\omega^2 I + M]}{\det\left[\omega^2 I + \bar{A}^2\right]}\,. \tag{B.12}$$

where

$$M := \bar{A}\left(\bar{A} - \frac{1}{\bar{d}}\bar{b}\bar{c}^T\right) = A^{-1}\left(A^{-1} - \frac{A^{-1}bc^TA^{-1}}{c^TA^{-1}b}\right).$$

Since A is Hurwitz, all the real eigenvalues of $\bar{A}^2 = A^{-2}$ are positive which implies that $\det[\omega^2 I + \bar{A}^2] \neq 0$ for all ω. Noting that $\det\left[\omega^2 I + \bar{A}^2\right] > 0$ for ω sufficiently large, it follows from continuity arguments that $\det[\omega^2 I + \bar{A}^2] > 0$ for all ω. Recalling that $\bar{d} > 0$ it follows from the above identity (B.12) that conditions (B.9) and (B.10) on $H(-j\omega^{-1})$ are respectively equivalent to

$$\det[\omega^2 I + M] > 0 \qquad \text{for all} \quad \omega \in \mathbb{R}, \quad \omega \neq 0$$
$$\lim_{\omega \to 0} \frac{1}{\omega^2}\det[\omega^2 I + M] > 0\,.$$

Since, $\det[\omega^2 I + M] > 0$ for large ω, the above conditions are equivalent to

$$\det[\lambda I - M] \neq 0 \qquad \text{for all} \quad \lambda \in \mathbb{R}, \quad \lambda < 0 \tag{B.13}$$
$$\lim_{\lambda \to 0} \frac{1}{\lambda}\det[\lambda I - M] \neq 0\,. \tag{B.14}$$

Condition (B.13) is equivalent to the requirement that M has no negative real eigenvalues. Since $Mb = 0$ and $b \neq 0$, the matrix M must have at least one zero eigenvalue; hence $\det[\lambda I - M] = \lambda q(\lambda)$ and all the other eigenvalues of M are given by the roots of the polynomial q. Thus condition (B.14) is equivalent to $q(0) \neq 0$, that is, zero is not a root of q. Thus (B.14) is equivalent to the requirement that M has only one eigenvalue at zero.

The equivalence between the first and third statement of the lemma follows from the SPR equivalence of $c^T(sI - A^{-1})^{-1}b$ and $c^T(sI - A)^{-1}b$ as stated in Lemma 2.

□

Comment : The literature contains spectral conditions for checking SPR of a strictly proper transfer function [29]; however, these conditions involve the eigenvalues of a $2n \times 2n$ Hamiltonian matrix. The conditions here involve a matrix of dimension $n \times n$.

B.2 Main result

In everything that follows, A is a real $n \times n$ matrix and g and h are real n-vectors. These results make use of the following observations.

A matrix $P = P^T > 0$ is a strict Lyapunov matrix for A, that is,

$$A^TP + PA < 0$$

if and only if P is a strict Lyapunov matrix for A^{-1}, that is,

$$A^{-T}P + PA^{-1} < 0$$

To see, this post- and pre-multiply the first inequality by A^{-1} and its transpose.

In a similar fashion one can also show that P is a (non-strict) Lyapunov matrix for A, that is,

$$A^T P + PA \leq 0$$

if and only if P is a (non-strict) Lyapunov matrix for A^{-1}, that is,

$$PA^{-1} + A^{-T} P \leq 0.$$

The proof of the main result requires the following KYP lemma.

Lemma 4. *[3] Suppose (A, b) is controllable and (A, c) is observable. Then, the following statements are equivalent.*

(i) The matrix A is Hurwitz and the transfer function $H(s) = c^T(sI - A)^{-1}b$ is SPR.

(ii) There exists a matrix $P = P^T > 0$ that satisfies the constrained Lyapunov inequality:

$$\begin{aligned} A^T P + PA &< 0 \\ Pb &= c\,. \end{aligned}$$

(iii) There exists a matrix $P = P^T > 0$ such that the following Lyapunov inequalities are satisfied:

$$\begin{aligned} A^T P + PA &< 0 \\ -\left(cb^T P + Pbc^T\right) &\leq 0. \end{aligned}$$

Comment : A discussion of a strictly positive real transfer function can be found in Narendra & Taylors book on Frequency domain stability criteria [22]. The assumption that (A,c) is observable ensures that P is positive definite in the theorem [1].

Theorem 2 (Main Theorem)**.** *Suppose that A is Hurwitz and all the eigenvalues of $A - gh^T$ have negative real part, except one, which is zero. Suppose also that (A, g) is controllable and (A, h) is observable. Then, there exists a matrix $P = P^T > 0$ such that*

$$A^T P + PA < 0 \tag{B.15}$$

$$(A - gh^T)^T P + P(A - gh^T) \leq 0 \tag{B.16}$$

if and only if the matrix product $A(A - gh^T)$ has no real negative eigenvalues and exactly one zero eigenvalue.

Proof : The proof consists of two parts. First we use an equivalence to show that the conditions on $A(A - gh^T)$ are sufficient for the existence of a Lyapunov matrix P with the required properties. We then show that these conditions are also necessary.

Sufficiency : Let $c = A^{-T}h$ and let b be a right eigenvector of $A - gh^T$ corresponding to the zero eigenvalue. Then $b \neq 0$, $Ab = gh^Tb = h^Tbg$ and $c^TAb = h^Tb$. Since A is Hurwitz, we must have $h^Tb \neq 0$, otherwise $Ab = 0$. Hence $c^TAb \neq 0$ and, without loss of generality, we assume that b is chosen so that $c^TAb = -1$. In this case,

$$g = -Ab \qquad \text{and} \qquad h^T = c^TA\,.$$

Controllability of (A, b) and observability of (A, c) follow from controllability of (A, g) and observability of (A, h), respectively.

Noting that

$$A_2 := A - gh^T = A - \frac{Abc^TA}{c^TAb}\,,$$

it follows from Theorem 1 that the conditions on AA_2 imply that the transfer function $c^T(sI - A)^{-1}b$ is SPR. Consequently, it follows from Lemma 4 that there exists a matrix $P = P^T > 0$ such that

$$A^TP + PA < 0 \tag{B.17}$$
$$Pb = c\,. \tag{B.18}$$

Pre- and post- multiplying the above inequality by A^{-T} and A^{-1} shows that this inequality is equivalent to

$$A^{-T}P + PA^{-1} < 0 \tag{B.19}$$

This last inequality and (B.18) imply that

$$\begin{bmatrix} A^{-T}P + PA^{-1} & Pb - c \\ b^TP - c^T & 0 \end{bmatrix} \leq 0 \tag{B.20}$$

that is,

$$\begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix} \leq 0\,.$$

Since $c^TAb = -1 \neq 0$,

$$\begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A - \frac{Abc^TA}{c^TAb} & -\frac{Ab}{c^TAb} \\ \frac{c^TA}{c^TAb} & \frac{1}{c^TAb} \end{bmatrix} = \begin{bmatrix} A - gh^T & -g \\ -h^T & -1 \end{bmatrix},$$

Post- and pre-multiplying inequality (B.20) by the above inverse and its transpose results in

$$\begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix}^{-T} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A^{-1} & b \\ -c^T & 0 \end{bmatrix}^{-1} \leq 0.$$

that is,

$$\begin{bmatrix} (A - gh^T)^TP + P(A - gh^T) & -Pg - h \\ -g^TP - h^T & -2 \end{bmatrix} \leq 0 \tag{B.21}$$

It immediately follows that for the above inequality to hold, we must have

$$(A - gh^T)^T P + P(A - gh^T) \leq 0. \tag{B.22}$$

Necessity : We first show that if there exists a matrix $P = P^T > 0$ satisfying conditions (B.15)-(B.16), then AA_2 cannot have a negative real eigenvalue. Note that the conditions on P are equivalent to

$$A^{-T}P + PA^{-1} < 0 \tag{B.23}$$

$$A_2^T P + PA_2 \leq 0 \tag{B.24}$$

Hence, for any $\gamma > 0$,

$$(A_2 + \gamma A^{-1})^T P + P(A_2 + \gamma A^{-1}) < 0\,.$$

Since $P = P^T > 0$ this Lyapunov inequality implies that $A_2 + \gamma A^{-1}$ must be Hurwitz and hence, non-singular. Thus $AA_2 + \gamma I$ is nonsingular for all $\gamma > 0$. This means that AA_2 cannot have a negative real eigenvalue.

We now show that AA_2 cannot have a zero eigenvalue whose multiplicity is greater that one. To this end introduce the matrix

$$\tilde{A}(k) = A_2 + kgh^T\,.$$

Then $A = \tilde{A}(1)$ and inequalities (B.15)-(B.16) hold if and only if

$$\tilde{A}(k)^T P + P\tilde{A}(k) < 0 \tag{B.25}$$

$$A_2^T P + PA_2 \leq 0 \tag{B.26}$$

hold for all k sufficiently close to one. As we have seen above, this implies that $A(k)A_2$ cannot have negative real eigenvalues for all k sufficiently close to one. We shall show that AA_2 having an eigenvalue at the origin whose multiplicity is greater than one contradicts this statement.

By assumption, A_2 has a single eigenvalue at zero; a corresponding eigenvector is the vector b. Clearly, b is also an eigenvector corresponding to a zero eigenvalue of $A(k)A_2$. Now choose any nonsingular matrix T whose first column is b. Then,

$$T^{-1}A(k)A_2T = \begin{pmatrix} 0 & * \\ 0 & S + krs^T \end{pmatrix} \tag{B.27}$$

and the eigenvalues of $A(k)A_2$ consist of zero and the eigenvalues of $S + krs^T$. Note that the matrix S must be invertible since

$$T^{-1}A_2^2T = T^{-1}A(0)A_2T = \begin{pmatrix} 0 & * \\ 0 & S \end{pmatrix}$$

and A_2^2 has only a single eigenvalue at zero. Now suppose that $AA_2 = A(1)A_2$ has an eigenvalue at the origin whose multiplicity is greater than one. Then $S + rd^T$ must have a eigenvalue at zero; hence, $\det\left[S + rs^T\right] = 0$. Since S is invertible,

$$\det\left[S + krs^T\right] = \det[S]\ \det\left[I + kS^{-1}rs^T\right] = \det[S]\,(1 + ks^TS^{-1}r)\,,$$

and we must have $1 + s^T S^{-1} r = 0$ which implies that $s^T S^{-1} r = -1$. Hence,

$$\det\left[S + krs^T\right] = \det[S](1-k)\,.$$

Suppose $\det[S] > 0$. Then,

$$\det\left[S + krs^T\right] < 0$$

for $k > 1$. Since $\det\left[S + krs^T\right]$ is the product of all the eigenvalues of $S + krs^T$ and complex eigenvalues occur in complex conjugate pairs, $S + krs^T$ must have at least one real negative eigenvalue when $k > 1$. This yields the contradiction that $A(k)A_2$ has a negative real eigenvalue when $k > 1$. The conclusion is the same for $\det[S] < 0$.

Bibliography

[1] A. Acikmese and M. Corless. Stability analysis with quadratic Lyapunov functions: Some necessary and sufficient multiplier conditions. *Systems and Control Letters*, 57:78–94, 2008.

[2] Z. Bai and W. Freund. Eigenvalue based characterisation and test for positive realness of scalar transfer functions. *IEEE Transactions on Automatic Control*, 45:2396–2402, 2000.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.

[4] C. Chien and P. Ioannou. Automatic vehicle following. In *Proceedings of the American Control Conference*, pages 1748–1752, 1992.

[5] K. C. Chu. Decentralized control of high-speed vehicular strings. *Transportation Science*, 8(4):361 – 384, 1974.

[6] C. Edwards and I. Postlethwaite. Anti-windup and bumpless transfer schemes. *Automatica*, 34(2):199–210, 1998.

[7] J. Eyre, D. Yanakiev, and I. Kanellakopoulos. A simplified framework for string stability analysis of automated vehicles. *Vehicle Systems Dynamic*, 30(5):375–405, 1998.

[8] J. Gomes da Silva and S. Tarbouriech. Antiwindup design with guaranteed regions of stability: An lmi-based approach. *IEEE Transactions on Automatic Control*, 50(1):106–111, 2005.

[9] G. Goodwin, S. Graebe, and M. Salgado. *Control System Design.* Prentice Hall, 2000.

[10] J. Hedrick, C. McMahon, and D. Swaroop. Vehicle modeling and control for automated highway systems. Technical report, University of California, Berkeley, 1993.

[11] R. E. Kalman. Lyapunov functions for the problem of Lur'e in automatic control. *Proceedings of the national academy of sciences*, 49(2):201–205, 1963.

[12] H. Khalil. *Nonlinear Systems.* Prentice Hall, third edition, 2001.

[13] M. Khatir and E. Davison. Bounded stability and eventual string stability of a large platoon of vehicles using non-identical controllers. In *Proceedings of the 43rd Conference on Decision & Control*, pages 1111–1116, 2004.

[14] M. Kothare, P. Campo, M. Morari, and C. Nett. A unified framework for the study of anti-windup designs. Technical Report CIT-CDS 93-011, California Institute of Technology, 1993.

[15] M. V. Kothare and M. Morari. Multiplier theory for stability analysis of anti-windup control systems. *Automatica*, 35:917–928, 1999.

[16] I. Lestas and G. Vinnicombe. Scalability in heterogeneous vehicle platoons. In *Proceedings of the American Control Conference*, pages 4678–4683, 2007.

[17] W. Levine and M. Athans. On the optimal error regulation of a string of moving vehicles. *IEEE Transactions on Automatic Control*, AC-11(3):355–361, 1966.

[18] T. Loquen, S. Tarbouriech, and C. Prieur. Stability analysis for reset systems with input saturation. In *Proceedings of the 46th IEEE Conference on Decision & Control*, pages 3272–3277, 2007.

[19] R. Middleton. Trade-offs in linear control system design. *Automatica*, 27(2):281–292, 1991.

[20] R. Middleton and J. Braslavsky. String instability in classes of linear time invariant formation control with limited communication range. Submitted to the IEEE Transactions on Automatic Control, 2008.

[21] G. Muscato, G. Nunarri, and L. Fortuna. Singular perturbation approximation of bounded real and positive real transfer matrices. In *Proceedings of the American Control Conference*, 1994.

[22] K. Narendra and J. Taylor. *Frequency Domain Criteria for Absolute Stability*. Academic Press, 1973.

[23] B. Noble and J. Daniel. *Applied Linear Algebra*. Prentice Hall, third edition, 1987.

[24] L. Peppard. String stability of relative-motion vehicle control systems. *IEEE Transactions on Automatic Control*, AC-19:579 – 581, 1974.

[25] B. Romanchuk and M. Smith. Incremental gain analysis of linear systems with bounded controls and its application to the antiwindup problem. In *Proceedings of the 35th Conference on Decision & Control*, pages 2942–2947, 1996.

[26] P. Seiler, A. Pant, and K. Hedrick. Disturbance propagation in vehicle strings. *IEEE Transactions on Automatic Control*, 49(10):1835 – 1841, 2004.

[27] S. Sheikholeslam and C. Desoer. Longitudinal control of a platoon of vehicles. In *Proceedings of the American Control Conference*, pages 291–297, 1990.

[28] R. Shorten, M. Corless, K. Wulff, S. Klinge, and R. Middleton. Quadratic stability and singular SISO switching systems. Submitted to the IEEE Transactions on Automatic Control, 2008.

[29] R. Shorten, P. Curran, K. Wulff, and E. Zeheb. A note on positive realness of transfer function matrices. Accepted for publication by IEEE Transactions on Automatic Control, 2008.

[30] R. Shorten and C. King. Spectral conditions for positive realness of SISO systems. *IEEE Transactions on Automatic Control*, 49:1875–1877, Oct 2004.

[31] D. Swaroop and J. Hedrick. String stability of interconnected systems. *IEEE Transactions on Automatic Control*, 41(3):349 – 357, 1996.

[32] D. Swaroop and D. Niemann. Some new results on the oscillatory behavior of impulse and step responses for linear time-invariant systems. In *Proceedings of the 35th Conference on Decision & Control*, 1996.

[33] A. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*. Springer, 1996.

[34] M. Vidyasagar. *Nonlinear Systems Analysis*. SIAM: Society for Industrial and Applied Mathematics, second edition, 2002.

[35] D. Yanakiev, J. Eyre, and I. Kanellakopoulos. Longitudinal control of heavy duty vehicles: Experimental evaluation. Technical report, Institute of Transportation Studies, University of California, 1998.

[36] D. Yanakiev and I. Kanellakopoulos. Variable time headway for string stability of automated heavy–duty vehicles. In *Proceedings of the 34th Conference on Decision & Control*, pages 4077 – 4081, 1995.

[37] K. Zhou, J. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1995.