Mathematical model of the robot
Part II

2020

The main purpose of this manual is to develop a simulator of the balancing robot that will then be used in the next labs for rapid debugging and prototyping purposes.

1 Derivation of the Equations of Motion

The Equations of Motion (EOM) are the heart of the simulation. Without them there is nothing to simulate. Furthermore, the EOM can be used for advanced filtering, for example in a Kalman filter.

The balancing robot should be modeled as in Figure 1 (check also the table for the notation on page **??**): the body of the robot can be simplified as a thin pole with its mass m_b concentrated only in the center of mass of the robot, depicted as a larger dot in figure 2. The center of mass is at a distance l_b from the center of the wheel. The wheel has a radius of l_w, and mass of m_w.

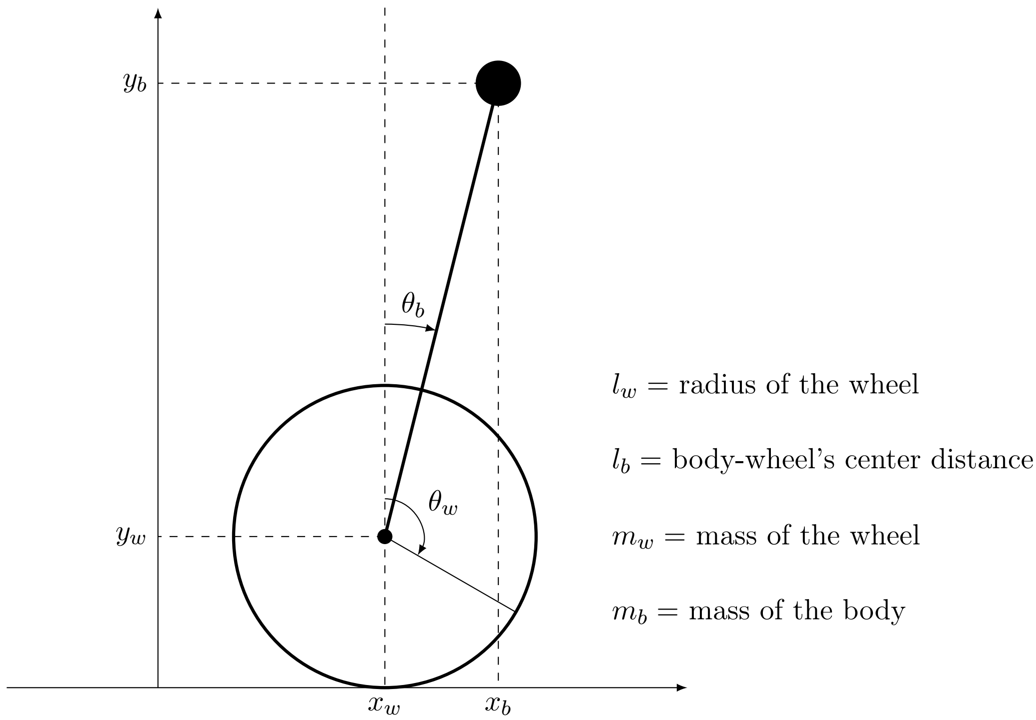
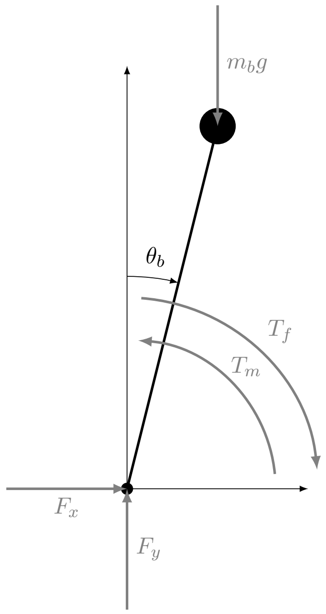


Figure 1: A simple schematic representation of a one-wheeled balancing robot where all the mass of robot (but the wheels') is concentrated in its center of mass.

The following assumptions should be made in order to simplify the problem:

1. The robot moves in a flat and horizontal environment, i.e., $\dot{y}_w = 0$ always.
2. The wheels never slip and the robot is never turned around by external factors, i.e., $x_w = l_w \theta_w$ always.
3. The aerodynamic frictions are negligible.
4. The inductance L_m and the motor viscous coefficient b_m are negligible.
5. The unique force that can be commanded is the torque applied by the motor to the wheel, and this torque is driven by the voltage that is applied to the motor.



F_y = vertical component of the tension with the wheel
F_x = horizontal component of the tension with the wheel
$m_b g$ = gravity for the body
T_m = motor torque
T_f = friction torque

Figure 2: Summary of the forces that apply to the body of the balancing robot.

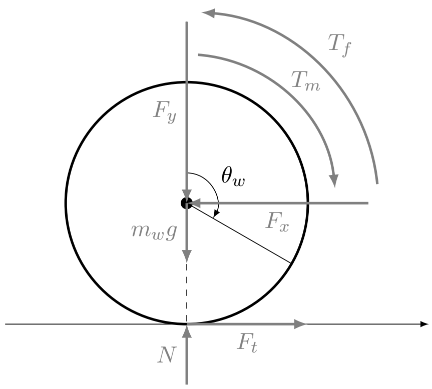
The Newton law for the linear movement of **the body** states that

$$m_b \ddot{x}_b = F_x \tag{1}$$
$$m_b \ddot{y}_b = F_y - m_b g \tag{2}$$

Notice that the gravity does not affect the x (horizontal) component since it is orthogonal to it. The Newton principle for the angular movement of the body (with rotational axis on the center of mass of the body) states that

$$I_b \ddot{\theta}_b = -T_m + T_f + F_y l_b \sin(\theta_b) - F_x l_b \cos(\theta_b) \tag{3}$$

for which the gravity does not affect the θ_b component since it does not lead to torque effects.



F_y = vertical component of the tension with the body
F_x = horizontal component of the tension with the body
$m_w g$ = gravity for the wheel
N = reaction of the plane
F_t = tractive force
T_m = motor torque
T_f = friction torque

Figure 3: Summary of the forces that apply to the wheel of the balancing robot.

The Newton law for the horizontal and vertical movements of the wheel are

$$m_w \ddot{x}_w = F_t - F_x \tag{4}$$
$$m_w \ddot{y}_w = N - m_w g - F_y = 0 \tag{5}$$

The Newton law for the angular movement of the wheel finally states that

$$I_w \ddot{\theta}_w = T_m - T_f - l_w F_t \tag{6}$$

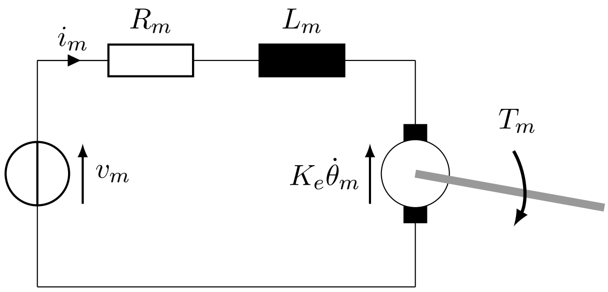


Figure 4: Schematic representation of a DC motor. Here, θ_m indicates the angle of the motor.

Analyzing the electrical circuit we get

$$L_m \frac{di_m}{dt} + R_m i_m = R_m i_m = v_m - e \tag{7}$$

where e is the back electromangetic force (emf), connected with the angular velocity of the motor through

$$e = K_e \left(\dot{\theta}_w - \dot{\theta}_b\right) = K_e \left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \tag{8}$$

where the first equality in (7) follows from the assumption that $L_m = 0$. Substituting (8) in (7) we then get

$$i_m = \frac{v_m}{R_m} - \frac{K_e}{R_m}\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \tag{9}$$

Consider then that the torque T_m on the wheel's shaft induced by an armature's current equal to i_m is $T_m = K_t i_m$, with K_t the motor torque constant. Substituting in (9) we eventually obtain

$$T_m = \frac{K_t}{R_m} v_m - \frac{K_e K_t}{R_m}\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \tag{10}$$

Since we have two motors, producing double the torque T_m, we replace the motor torque by

$$\hat{T}_m = 2T_m = \frac{2K_t}{R_m} v_m - \frac{2K_e K_t}{R_m}\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right)$$

Remark 1.1

Be careful at the quantities that are considered in the various tasks: pay specially attention at the subscripts, and remember that w means "wheel", b means "body", m means "motor".

Remark 1.2

In order to rewrite F_t, F_x and F_y the Newton's laws for the linear movements of the wheel and of the body can be used. If everything goes as expected then there will be the need for simplifying the quantity "$\ddot{y}_b \sin(\theta_b) - \ddot{x}_b \cos(\theta_b)$". The following equivalences can be exploited for the simplification:

$$\begin{cases} x_b &= x_w + l_b \sin(\theta_b) \\ \dot{x}_b &= \dot{x}_w + \dot{\theta}_b l_b \cos(\theta_b) \\ \ddot{x}_b &= \ddot{x}_w + \ddot{\theta}_b l_b \cos(\theta_b) - \dot{\theta}_b^2 l_b \sin(\theta_b) \\ y_b &= y_w + l_b \cos(\theta_b) \\ \dot{y}_b &= \dot{y}_w - \dot{\theta}_b l_b \sin(\theta_b) = -\dot{\theta}_b l_b \sin(\theta_b) \\ \ddot{y}_b &= -\ddot{\theta}_b l_b \sin(\theta_b) - \dot{\theta}_b^2 l_b \cos(\theta_b) \end{cases} \tag{11}$$

Eliminating F_t, F_x and F_y from (3) and from (6) leads to two different equations of motion.

Equation 1: To eliminate F_x and F_y from (3) we can exploit (1) and (2) to obtain

$$I_b \ddot{\theta}_b = -\hat{T}_m + T_f + m_b l_b g \sin(\theta_b) + m_b l_b \ddot{y}_b \sin(\theta_b) - m_b l_b \ddot{x}_b \cos(\theta_b). \tag{12}$$

We don't like too much this expression, since it contains x_b and y_b terms. So we now aim to rewrite $\ddot{y}_b \sin(\theta_b) - \ddot{x}_b \cos(\theta_b)$ in a different way. Considering then Figure 1 on page ii, it follows that $\ddot{x}_b$ and $\ddot{y}_b$ are linked to $\ddot{x}_w$ and $\ddot{\theta}_b$ as in (11). Thus

$$\begin{aligned}
&+\ddot{y}_b \sin(\theta_b) - \ddot{x}_b \cos(\theta_b) = \\
&\quad \Big(-\ddot{\theta}_b l_b \sin(\theta_b) - \dot{\theta}_b^2 l_b \cos(\theta_b)\Big) \sin(\theta_b) - \Big(\ddot{x}_w + \ddot{\theta}_b l_b \cos(\theta_b) - \dot{\theta}_b l_b \sin(\theta_b)\Big) \cos(\theta_b) = \\
&\quad -\ddot{\theta}_b l_b \sin^2(\theta_b) - \dot{\theta}_b^2 l_b \cos(\theta_b) \sin(\theta_b) - \ddot{x}_w \cos(\theta_b) - \ddot{\theta}_b l_b \cos^2(\theta_b) + \dot{\theta}_b^2 l_b \sin(\theta_b) \cos(\theta_b) = \\
&\quad -\ddot{\theta}_b l_b - \ddot{x}_w \cos(\theta_b)
\end{aligned} \tag{13}$$

Plugging into (12) we then obtain

$$I_b \ddot{\theta}_b = -\hat{T}_m + T_f + m_b l_b g \sin(\theta_b) - m_b l_b^2 \ddot{\theta}_b - m_b l_b \ddot{x}_w \cos(\theta_b) \tag{14}$$

and thus, rearranging,

$$\boxed{\left(I_b + m_b l_b^2\right) \ddot{\theta}_b = +m_b l_b g \sin(\theta_b) - m_b l_b \ddot{x}_w \cos(\theta_b) - \hat{T}_m + T_f} \tag{15}$$

Equation 2: Plugging (4) into (6), and using $\ddot{\theta}_w = \ddot{x}_w / l_w$ leads to

$$\frac{I_w}{l_w} \ddot{x}_w = \hat{T}_m - T_f - l_w F_x - l_w m_w \ddot{x}_w. \tag{16}$$

To eliminate F_x we then combine (1) and (11) into

$$F_x = m_b \ddot{x}_w + m_b l_b \ddot{\theta}_b \cos(\theta_b) - m_b l_b \dot{\theta}_b^2 \sin(\theta_b). \tag{17}$$

Plugging this into (16) we then obtain

$$\frac{I_w}{l_w} \ddot{x}_w = \hat{T}_m - T_f - l_w \Big(m_b \ddot{x}_w + m_b l_b \ddot{\theta}_b \cos(\theta_b) - m_b l_b \dot{\theta}_b^2 \sin(\theta_b)\Big) - l_w m_w \ddot{x}_w \tag{18}$$

and, rearranging,

$$\boxed{\left(\frac{I_w}{l_w} + l_w m_b + l_w m_w\right) \ddot{x}_w = -m_b l_b l_w \ddot{\theta}_b \cos(\theta_b) + m_b l_b l_w \dot{\theta}_b^2 \sin(\theta_b) - T_f + \hat{T}_m.} \tag{19}$$

2 Linearization of the Equations of Motion

Since the goal is to design a controller, if the mathematical model is too complicated, we will have difficulties in the design. We know a lot about controllers when the system is linear. Therefore, it is better if the EOM are linearized.

Since the operation of the robot will be around the equilibrium, the linearization point must be the equilibrium $\theta_b = 0$. Therefore,

- $\sin(\theta_b) \approx \theta_b$;
- $\ddot{x}_w \cos(\theta_b) \approx \ddot{x}_w$;
- $\ddot{\theta}_b \cos(\theta_b) \approx \ddot{\theta}_b$.

As for $\dot{\theta}_b^2 \sin(\theta_b)$, the suggestion is to assume negligible centripetal forces (i.e., small body angle velocities), so that we can say $\dot{\theta}_b^2 \approx 0$. Thus, the linearized EOM are

$$\begin{cases} \left(I_b + m_b l_b^2\right) \ddot{\theta}_b &= +m_b l_b g \theta_b - m_b l_b \ddot{x}_w - \dfrac{2K_t}{R_m} v_m + \left(\dfrac{2K_e K_t}{R_m} + b_f\right)\left(\dfrac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \\ \left(\dfrac{I_w}{l_w} + l_w m_b + l_w m_w\right) \ddot{x}_w &= -m_b l_b l_w \ddot{\theta}_b + \dfrac{2K_t}{R_m} v_m - \left(\dfrac{2K_e K_t}{R_m} + b_f\right)\left(\dfrac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \end{cases} \tag{20}$$