Decentralized control for l_2 weak string stability of vehicle platoon

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Abstract: In this paper we present a method of bidirectional control of a vehicle string that achieves l_2 weak string stability. Previous work required short-range communications between vehicles within the string. Here, we utilise some recent results on integral action to remove the requirement of communications altogether.

Keywords: Multi-vehicle systems, Lagrangian and Hamiltonian systems, Disturbance rejection

1. INTRODUCTION

Consider a string or platoon of N vehicles where each car is driven by an automatic controller aiming to keep a prescribed distance towards the surrounding vehicles, for instance its direct predecessor and follower. In absence of a direct predecessor, the first vehicle tracks a given reference signal. String stability is concerned with how the behaviour of the platoon changes as N grows (Swaroop and Hedrick, 1996). In particular, it is concerned with the bounds of local errors, within a string of arbitrary length, as the string responds to both initial conditions and external disturbances. If a bound for the local error signals can be found, which is independent of N, the system is considered “string stable”.

In the case that the local controller only considers information of preceding vehicles, the string is called “unidirectional”, and “bidirectional” otherwise. (Seiler et al., 2004) showed that string stability of unidirectional strings cannot be achieved in linear vehicle strings with double integrators in the open loop, local information only and tight spacing. Solutions to ensure string stability include using a time headway spacing policy (Chien and Ioannou, 1992; Middleton and Braslavsky, 2010), local controllers that depend on the position within the string (Khatir and Davison, 2004), or, allowing inter-vehicle communication to propagate additional information such as the lead vehicle’s states or the reference signal through the string, (Barooah et al., 2009).

In this paper, we focus on a possibly nonlinear, symmetric, bidirectional control scheme. It was shown by (Barooah and Hespanha, 2005) that linear, bidirectional control scheme cannot be used to achieve string stability. As this class is contained within the class of controllers studied here, we rely on a weaker notion of l_2 weak string stability, which can be achieved for bidirectional vehicle strings with tight spacing, (Knorn et al., 2014). This weaker notion considers the behaviour of the string when subjected to a disturbance of bounded l_2 function norm. The results were extended to analyse the effects of measurement errors in (Knorn et al., 2015).

Similar to (Knorn et al., 2014), we design a controller that achieves l_2 weak string stability in two steps. The primary controller used has a natural interpretation as fictitious springs and dampers acting between the vehicles. As such, the system is naturally modelled as a port-Hamiltonian (pH) system. Although this first control loop has some interesting string stability properties, does not satisfy all the control objective requirements. In particular, the space between vehicles in steady state increase with the number of vehicles in the string. Surprisingly, this problem can be easily corrected by adding an integral action control, which can be performed using the approaches in the pH framework using the methods in (Donaire and Junco, 2009; Ortega and Romero, 2012; Ferguson et al., 2017).

Previously, the integral action scheme of (Donaire and Junco, 2009) was used in (Knorn et al., 2014). For each vehicle within the string, this scheme required knowledge of the distance errors between the vehicle’s direct predecessor and follower as well as their distance errors between their predecessor or follower. That is, the information of four neighbour vehicles was required by each vehicle. As these states cannot be locally measured, this information would be obtained via communications between neighbouring vehicles.

In this paper, we shown that the communication requirements in (Knorn et al., 2014) can be relaxed by instead using the integral action scheme of (Ferguson et al., 2017). It is shown that a particular implementation of this integral action scheme only requires knowledge of a vehicle’s direct predecessor and follower, which can be locally measured. Thus, this alternate approach removes the need for communications altogether. Furthermore, numerical simulations show that the transients do not exhibit oscillatory behaviour which was evident in previous work.

2. NOTATION AND BACKGROUND

2.1 Notation

The operator $\mathrm{col}(\cdot)$ defines a column vector of its arguments, $\mathrm{col}(x_1, x_2, \cdots, x_n) = [x_1 \; x_2 \; \cdots \; x_n]^\top$, whereas $\mathrm{diag}(\cdot)$ defines a diagonal matrix of its arguments. For any given vector $x = \mathrm{col}(x_1, x_2, \cdots, x_n)$, the l_2 vector norm is denoted $|x|_2 = \sqrt{x^\top x}$, the l_2 vector function norm is defined as $||x(t)||_2 = \sqrt{\int_0^\infty x^\top(t)x(t)dt}$ and the l_∞ function norm is defined as $||x(t)||_\infty = \sup_{t\geq 0} |x(t)|_2$. Given a scalar function $g(x)$, the total gradient is defined as $\nabla g = \left[\frac{\partial g}{\partial x_1} \; \frac{\partial g}{\partial x_2} \; \cdots \; \frac{\partial g}{\partial x_n}\right]^\top$, where as a partial gradient is denoted by subscript, $\nabla_{x_i} g = \frac{\partial g}{\partial x_i}$. For some matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of A respectively. Further, the dimension of a vector x is denoted $\dim(x)$.

2.2 Port-Hamiltonian systems

Port-Hamiltonian (pH) systems are dynamic systems of the form

$$\begin{aligned} \dot{x} &= (J - R)\nabla\mathcal{H} + Gu \\ y &= G^\top \nabla\mathcal{H} \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^p$ is the state vector, $J = -J^\top$ is the power preserving interconnection structure, $R = R^\top \geq 0$ is the dissipation structure, $\mathcal{H} \geq 0$ is the Hamiltonian corresponding to the total system energy, $u \in \mathbb{R}^m$ is the input to the system, G is the input mapping matrix and $y \in \mathbb{R}^m$ is the passive output corresponding to input u. It can be verified that this class of system is passive with storage function $\mathcal{H}$, input u and output y (van der Schaft and Jeltsema, 2014).

2.3 l_2 weak string stability

The study of string stability is concerned with the stability properties of the strings state vector, independent of the string length N. In this paper, we focus on the property of l_2 weak string stability.

Definition 1. [Knorn et al. (2014)] Consider a system described by $\dot{x} = g(x, d)$ with states $x \in \mathbb{R}^P$, disturbances $d \in \mathbb{R}^M$ and $g : \mathbb{R}^P \times \mathbb{R}^M \to \mathbb{R}^P$ satisfying $g(x^*, 0) = 0$. The equilibrium x^* is l_2 weakly string stable with respect to disturbances $d(\cdot)$, if given any $\epsilon > 0$, there exists $\delta_1(\epsilon) > 0$ and $\delta_2(\epsilon) > 0$ (independent of P) such that

$$|x(0) - x^*| < \delta_1(\epsilon) \text{ and } ||d(\cdot)||_2 < \delta_2(\epsilon) \tag{2}$$

implies

$$||x(t) - x^*||_\infty = \sup_{t\geq 0} |x(t) - x^*| < \epsilon \;\forall P \geq 1. \tag{3}$$

2.4 Integral action

For pH systems, a method for the addition of integral action was presented in (Donaire and Junco, 2009). More recently, an alternate method for the addition of integral action to a class of pH systems was presented in (Ferguson et al., 2017). We will now review this result:

Consider the class pH systems described by

$$\begin{aligned} \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix} J_1 - R_1 & J_{12} \\ -J_{12}^\top & J_2 - R_2 \end{bmatrix} \begin{bmatrix}\nabla_{x_1}\mathcal{H} \\ \nabla_{x_2}\mathcal{H}\end{bmatrix} + \begin{bmatrix} I \\ 0\end{bmatrix} u - \begin{bmatrix} I \\ 0\end{bmatrix} d_1 \\ y &= \nabla_{x_1}\mathcal{H}, \end{aligned} \tag{4}$$

where $\dim(x_1) = \dim(x_2) = n$, $J_1 = -J_1^\top$, $J_2 = -J_2^\top$, $R_1 = R_1^\top > 0$, $R_2 = R_2^\top \geq 0$ and $\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}$ is strictly convex, smooth and has a minimum at (x_1^*, x_2^*). $d_1 \in \mathbb{R}^n$ is a constant, matched disturbance to the system.

For the purpose of disturbance rejection, consider the control law

$$\begin{aligned} u &= (J_1 - R_1)\nabla_{x_1}\mathcal{H}_c(x_1 - \zeta) \\ \dot{\zeta} &= J_{12}\nabla_{x_2}\mathcal{H}, \end{aligned} \tag{5}$$

where $\mathcal{H}_c$ is a strictly convex function in variables $z := x_1 - \zeta$ such that $\nabla_z \mathcal{H}_c(z) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. The system (4), together with the control law (5), results in the closed loop dynamics

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{\zeta}\end{bmatrix} = \begin{bmatrix} J_1 - R_1 & J_{12} & 0 \\ -J_{12}^\top & J_2 - R_2 & -J_{12}^\top \\ 0 & J_{12} & 0 \end{bmatrix} \begin{bmatrix}\nabla_{x_1}\mathcal{H}_{cl} \\ \nabla_{x_2}\mathcal{H}_{cl} \\ \nabla_{\zeta}\mathcal{H}_{cl}\end{bmatrix} - \begin{bmatrix} I \\ 0 \\ 0\end{bmatrix} d_1, \tag{6}$$

where $\mathcal{H}_{cl} = \mathcal{H} + \mathcal{H}_c$.

The closed loop of the system (6) has the unique equilibrium point

$$\begin{bmatrix}\nabla_{x_1}\mathcal{H}_{cl}^* \\ \nabla_{x_2}\mathcal{H}_{cl}^* \\ \nabla_{\zeta}\mathcal{H}_{cl}^*\end{bmatrix} = \begin{bmatrix} (J_1 - R_1)^{-1}d_1 \\ 0 \\ -(J_1 - R_1)^{-1}d_1 \end{bmatrix}, \tag{7}$$

which corresponds to

$$\begin{bmatrix}\nabla_{x_1}\mathcal{H}^* \\ \nabla_{x_2}\mathcal{H}^* \\ \nabla_{z}\mathcal{H}_{c}^*\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (J_1 - R_1)^{-1}d_1 \end{bmatrix}. \tag{8}$$

Proposition 2. [Ferguson et al. (2017)] Consider system (4) in closed loop with the controller (5). If (4) satisfies:

- $J_1 - R_1$ is constant,
- J_{12} is full rank,
- $\nabla_{x_1 x_2}\mathcal{H} = 0$,
- $\mathcal{H}$ is strongly convex,

then, the equilibrium corresponding to the gradient (7) is globally asymptotically stable.

Proof. [sketch] Take

$$\mathcal{W} = \mathcal{H}_{cl}(w) - [w - w^*]^\top \nabla\mathcal{H}_{cl}^* - \mathcal{H}_{cl}(w^*) \tag{9}$$

as a lyapunov candidate, where $w = \mathrm{col}(x_1, x_2, \zeta)$. The time derivative of (9) can be computed along the trajectories of the system:

$$\begin{aligned} \dot{\mathcal{W}} &= \{\nabla\mathcal{H}_{cl} - \nabla\mathcal{H}_{cl}^*\}^\top \dot{w} \\ &\leq - \begin{bmatrix}\nabla_{x_1}\mathcal{H}_{cl} \\ \nabla_{x_1}\mathcal{H}_{cl}^*\end{bmatrix}^\top \begin{bmatrix} I \\ -I\end{bmatrix} R_1 \,[I \; -I] \begin{bmatrix}\nabla_{x_1}\mathcal{H}_{cl} \\ \nabla_{x_1}\mathcal{H}_{cl}^*\end{bmatrix}, \end{aligned} \tag{10}$$

which demonstrates lyapunov stability. Asymptotic stability follows from invariance arguments.

3. PROBLEM FORMULATION AND PREVIOUS WORK

3.1 Problem formulation

We consider a string of N vehicles travelling in 1 dimension. Let m_i, q_i and p_i denote the mass, position and momentum of the i^{th} vehicle respectively. The distance between the i^{th} vehicle and the desired distance from its predecessor, the $(i-1)^{th}$ vehicle, is denoted by $\Delta_i := q_{i-1} + l_i - q_i$, where l_i is the desired steady-state distance between the two vehicles. Each vehicle has local force control F_i, satisfying $\dot{p}_i = F_i$. As was done in (Knorn et al., 2014), we model the vehicle string as the port-Hamiltonian (pH) system

$$\begin{aligned}\begin{bmatrix}\dot{p}\\ \dot{\Delta}\end{bmatrix} &= \begin{bmatrix}0 & S^\top\\ -S & 0\end{bmatrix}\begin{bmatrix}M^{-1}p\\ 0\end{bmatrix} + \begin{bmatrix}I\\ 0\end{bmatrix}F + \begin{bmatrix}d(t)\\ e_1 v_0\end{bmatrix}\\ y &= M^{-1}p\\ \mathcal{H} &= \frac{1}{2}p^\top M^{-1}p,\end{aligned} \tag{11}$$

where $M = \mathrm{diag}(m_1, \ldots, m_N), p = \mathrm{col}(p_1, \ldots, p_N), \Delta = \mathrm{col}(\Delta_1, \ldots, \Delta_N), F = \mathrm{col}(F_1, \ldots, F_N)$, e_1 is the 1^{st} standard basis vector and v_0 is the desired string velocity, known only by the first vehicle, and the matrix S is defined by

$$S = \begin{bmatrix}1 & 0 & 0 & & 0 & 0\\ -1 & 1 & 0 & \cdots & 0 & 0\\ 0 & -1 & 1 & & 0 & 0\\ & \vdots & & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & 1\end{bmatrix}. \tag{12}$$

The vehicle string (11) is subject to an l_2 bounded force disturbance, $d(t)$. That is, there exists some constant $\bar{d} < \infty$ such that $||d(t)||_2 \leq \bar{d}$. The objective is to design a dynamic controller $F = F(p, \Delta, \zeta)$ such that (11) is l_2 weakly string stable about the equilibrium $(p^*, \Delta^*) = (Mv_0, 0)$. Furthermore, we require that such a control law utilises only locally available information.

3.2 Previous work

The problem of achieving l_2 weak string stability of (11) was investigated by (Knorn et al., 2014). The following control law was proposed:

$$F = -(B + D)M^{-1}p + e_1 D_1 v_0 + S^\top f^s(\Delta) + \tilde{F}, \tag{13}$$

where $B = \mathrm{diag}(b_1, \cdots, b_n) > 0$ is the matrix of absolute damping coefficients between each vehicle and the ground, $f^s = \mathrm{col}(f_1^s(\Delta_1), \cdots, f_N^s(\Delta_N))$ is the vector of forces due to (possibly non-linear) virtual springs acting between the vehicles and

$$D = \begin{bmatrix}D_1 + D_2 & -D_2 & 0 & & 0 & 0\\ -D_2 & D_2 + D_3 & -D_3 & \cdots & 0 & 0\\ 0 & -D_3 & \ddots & & 0 & 0\\ \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & D_{N-1} + D_N & -D_N\\ 0 & 0 & 0 & & -D_N & D_N\end{bmatrix}, \tag{14}$$

where D_i is the damping coefficient of a linear damper acting between the i^{th} and $(i-1)^{th}$ vehicles.

Notice that the term $(B + D)$ is tridiagonal and $M^{-1}p = \dot{q}$ is the column vector of velocities. Similarly, $S^\top$ is upper bidiagonal and $f_i^s(\Delta)$ depends only on Δ_i. Thus, the i^{th} component of (13) is

$$\begin{aligned}F_i = D_i(\dot{q}_{i-1} - \dot{q}_i) - D_{i+1}(\dot{q}_i - \dot{q}_{i+1}) - b_i\dot{q}_i\\ + f_i^s(q_i - q_{i-1}) - f_i^s(q_{i+1} - q_i) + \tilde{F}_i.\end{aligned} \tag{15}$$

In the case that $\tilde{F}_i = 0$, the control scheme of the i^{th} vehicle only requires state information of its direct predecessor and follower, which can be locally measured using appropriate sensors. For example Doppler radar could be used to obtain relative velocity and distance measurements.

The string system (11), subject to the control law (13), results in the closed loop dynamics

$$\begin{aligned}\begin{bmatrix}\dot{p}\\ \dot{\Delta}\end{bmatrix} &= \begin{bmatrix}-(B + D) & S^\top\\ -S & 0\end{bmatrix}\nabla\mathcal{H}_{cl}(p, \Delta) + \begin{bmatrix}\tilde{F}\\ 0\end{bmatrix} + \begin{bmatrix}d(t)\\ 0\end{bmatrix}\\ y &= M^{-1}p - \underline{1}v_0\\ \mathcal{H}_{cl} &= \frac{1}{2}(p - M\underline{1}v_0)^\top M^{-1}(p - M\underline{1}v_0)\\ &\quad + \sum_{i=1}^{N}\int_{\Delta_i^*}^{\Delta_i}\left(f_i^s(w) - \sum_{k=i}^{N} b_k v_0\right)dw,\end{aligned} \tag{16}$$

where $\tilde{F}$ is an additional input for further control design.

Lemma 3. [Knorn et al. (2014)] Consider the string system (11) in closed loop with the control law (13) and where the initial spatial deviation $\Delta_i(0)$, the equilibrium state Δ_i^* and all values of Δ^* in between are in the domain of definition of $f_i^s(\Delta_i)$ for all i. Consider further that $f_i^s(\Delta_i)$ is a strictly monotonically increasing function satisfying $f_i^s(0) = 0$. Then, the equilibrium $(p^*, \Delta^*) = (M\underline{1}v_0, {f^s}^{-1}(S^{-\top}B\underline{1}v_0))$ is globally asymptotically stable in the absence of disturbances, i.e. $d = 0$.

As demonstrated by Lemma 3, the absolute damping term B causes the equilibrium point Δ^* to be away from zero. However, B must necessarily be non zero to achieve l_2 weak string stability. To correct the steady-state error of Δ^*, integral action was introduced via the additional input $\tilde{F}$.

In (Knorn et al., 2014), the integral action followed the scheme of Donaire and Junco (2009) and took the form:

$$\begin{aligned}\tilde{F} &= -A_p M^{-1}p + MKS^\top f^s(\Delta) - \underbrace{(B + R + A_p)K}_{Y} z_3\\ \dot{z}_3 &= -S^\top f^s(\Delta),\end{aligned} \tag{17}$$

where A_p, K are constant, diagonal, positive matrices to be used as tuning parameters. Notice that the matrix Y is tridiagonal, which implies that the integral control of the i^{th} vehicle, $\tilde{F}_i$, will depend on the integral state of both its predecessor and follower. Thus, implementation of such a scheme requires communications between neighbouring vehicles. In the remainder of this paper, we present an alternate implementation of integral action that avoids the need for communications between neighbouring vehicles

4. MAIN RESULT

The dynamics (16) can be rewritten as a disturbed pH system by removing the effects of the absolute damping B from the closed loop Hamiltonian:

$$\begin{aligned}
\begin{bmatrix}\dot{p}\\ \dot{\Delta}\end{bmatrix} &= \begin{bmatrix}-(B+D) & S^\top \\ -S & 0\end{bmatrix}\nabla\tilde{\mathcal{H}}_{cl}(p,\Delta) + \begin{bmatrix}\tilde{F}\\ 0\end{bmatrix} - \begin{bmatrix}B\underline{1}v_0 - d(t)\\ 0\end{bmatrix}\\
y &= M^{-1}p - \underline{1}v_0\\
\tilde{\mathcal{H}}_{cl} &= \frac{1}{2}(p - M\underline{1}v_0)^\top M^{-1}(p - M\underline{1}v_0) + \sum_{i=1}^{N}\int_{\Delta_i^*}^{\Delta_i} f_i^s(v)dv.
\end{aligned} \tag{18}$$

In the case $d(t) = 0$, (18) is of the form considered in Proposition 2. As such, we propose to implement an integral action controller of the form (5) to remove the effects of the disturbance $B\underline{1}v_0$.

Taking

$$\mathcal{H}_c = \frac{1}{2}k(p-\zeta)^\top(B+D)^{-1}(p-\zeta), \tag{19}$$

where $k > 0$ is a scalar control variable, the integral action control law (5), applied to (18) resolves to

$$\begin{aligned}
\tilde{F} &= -k(p-\zeta)\\
\dot{\zeta} &= S^\top f^s(\Delta).
\end{aligned} \tag{20}$$

For each vehicle within the string, the integral control law is:

$$\begin{aligned}
\tilde{F}_i &= -k(p_i - \zeta_i)\\
\dot{\zeta}_i &= f_i^s(\Delta_i) - f_{i+1}^s(\Delta_{i+1}),
\end{aligned} \tag{21}$$

where p_i is the i^{th} vehicles momenta, ζ_i is the i^{th} vehicles integral state and $f_i^s(\Delta_i), f_{i+1}^s(\Delta_{i+1})$ are the forces due to the fictitious springs acting between the i^{th} vehicles predecessor and follower respectively. Thus the integral control scheme only requires information that can be measured locally.

The closed loop dynamics can be written as the disturbed, pH system

$$\begin{aligned}
\begin{bmatrix}\dot{p}\\ \dot{\Delta}\\ \dot{\zeta}\end{bmatrix} &= \underbrace{\begin{bmatrix}-(B+D) & S^\top & 0\\ -S & 0 & -S\\ 0 & S^\top & 0\end{bmatrix}}_{:=A}\nabla\mathcal{H}_i(p,\Delta,\zeta) - \begin{bmatrix}B\underline{1}v_0 - d(t)\\ 0\\ 0\end{bmatrix}\\
\mathcal{H}_i &= \tilde{H}_{cl} + \mathcal{H}_c.
\end{aligned} \tag{22}$$

Now presume that the disturbance $d(t)$ is comprised of a constant component d_c and a dynamic component $d_d(t)$. That is, $d(t) = d_c + d_d(t)$. It can be verified that in the case of constant disturbances only ($d_d(t) = 0$), the equilibrium of (22) satisfies

$$\begin{bmatrix}\nabla_p\mathcal{H}_i^*\\ \nabla_\Delta\mathcal{H}_i^*\\ \nabla_\zeta\mathcal{H}_i^*\end{bmatrix} = \begin{bmatrix}-(B+D)^{-1}(B\underline{1}v_0 - d_c)\\ 0\\ (B+D)^{-1}(B\underline{1}v_0 - d_c)\end{bmatrix}, \tag{23}$$

which corresponds to the equilibrium

$$(p^*, \Delta^*, \zeta^*) = \left(M\underline{1}v_0, 0, M\underline{1}v_0 + \frac{1}{k}\left(B\underline{1}v_0 - d_c\right)\right). \tag{24}$$

We use this equilibrium to construct a Lyapunov candidate for the system (22).

Lemma 4. Consider the function

$$\mathcal{W} = \mathcal{H}_i(w) - [w - w^*]^\top\nabla\mathcal{H}_i^* - \mathcal{H}_i(w^*), \tag{25}$$

where $w = \mathrm{col}(p, \Delta, \zeta)$ and $\nabla\mathcal{H}_i^*$ and w^* refer to the equilibrium in (23) and (24) respectively. The function $\mathcal{W}$ can be expressed as

$$\begin{aligned}
\mathcal{W} &= \frac{1}{2}(p - M\underline{1}v_0)^\top M^{-1}(p - M\underline{1}v_0) + \sum_{i=1}^{N}\int_{\Delta_i^*}^{\Delta_i} f_i^s(v)dv\\
&\quad + \frac{1}{2k}[k(p-\zeta) + (B\underline{1}v_0 - d_c)]^\top(B+D)^{-1}\\
&\qquad \times [k(p-\zeta) + (B\underline{1}v_0 - d_c)].
\end{aligned} \tag{26}$$

Proof. Recall that $\mathcal{H}_i$ is the sum of $\tilde{H}_{cl}$ and $\mathcal{H}_c$, given by (18) and (19) respectively. Similarly, w^* and $\nabla_w\mathcal{H}_i^*$ are given by (24) and (23) respectively.

The terms $\mathcal{H}_i(w^*)$ and $[w - w^*]^\top\nabla\mathcal{H}_i^*$ are evaluated as

$$\mathcal{H}_i(w^*) = \frac{1}{2k}\left(B\underline{1}v_0 - d_c\right)^\top(B+D)^{-1}\left(B\underline{1}v_0 - d_c\right) \tag{27}$$

and

$$\begin{aligned}
&[w - w^*]^\top\nabla\mathcal{H}_i^*\\
&\quad = -[p - \zeta + \frac{1}{k}(B\underline{1}v_0 - d_c)]^\top(B+D)^{-1}(B\underline{1}v_0 - d_c)\\
&\quad = -(p-\zeta)^\top(B+D)^{-1}(B\underline{1}v_0 - d_c)\\
&\qquad - \frac{1}{k}(B\underline{1}v_0 - d_c)^\top(B+D)^{-1}(B\underline{1}v_0 - d_c).
\end{aligned} \tag{28}$$

Using (27) and (28), (25) can be evaluated:

$$\begin{aligned}
\mathcal{W} &= \mathcal{H}_i(w) - [w - w^*]^\top\nabla\mathcal{H}_i^* - \mathcal{H}_i(w^*)\\
&= \frac{1}{2}(p - M\underline{1}v_0)^\top M^{-1}(p - M\underline{1}v_0) + \sum_{i=1}^{N}\int_{\Delta_i^*}^{\Delta_i} f_i^s(v)dv\\
&\quad + \frac{1}{2}k(p-\zeta)^\top(B+D)^{-1}(p-\zeta)\\
&\quad + (p-\zeta)^\top(B+D)^{-1}(B\underline{1}v_0 - d_c)\\
&\quad + \frac{1}{2k}(B\underline{1}v_0 - d_c)^\top(B+D)^{-1}(B\underline{1}v_0 - d_c)\\
&= \frac{1}{2}(p - M\underline{1}v_0)^\top M^{-1}(p - M\underline{1}v_0) + \sum_{i=1}^{N}\int_{\Delta_i^*}^{\Delta_i} f_i^s(v)dv\\
&\quad + \frac{1}{2k}[k(p-\zeta) + (B\underline{1}v_0 - d_c)]^\top(B+D)^{-1}\\
&\qquad \times [k(p-\zeta) + (B\underline{1}v_0 - d_c)].
\end{aligned} \tag{29}$$

The behaviour of (22) is considered in the following Proposition.

Proposition 5. Assume the disturbance $d(t)$ includes a constant component d_c and a dynamical component $d_d(t)$ such that $d(t) = d_c + d_d(t)$ and there exists a constant $\bar{d} < \infty$ satisfying $||d_d(t)||_2 \leq \bar{d}$. Consider the string system (11) with a reference signal with constant velocity v_0, disturbance $d(t)$ in closed loop with a controller obtained by adding the symmetric, bidirectional control (13), together with the integral control (20). Then,

1. For $d_d(t) = 0$, the equilibrium (23) is asymptotically stable.
2. The closed loop (22) is passive with input $d_d(t)$, output $-[\nabla_p\mathcal{H}_i - \nabla_p\mathcal{H}_i^*]$ and storage function (25).

(3) the system is l_2 weakly string stable with respect to the time varying disturbance $d_d(t)$.

Proof.

(1) The proof follows from direct application of Proposition 2.

(2) Let $E = [I\ 0\ 0]^\top$. Taking (25) as the storage function, we compute the time derivative:

$$\begin{aligned}
\dot{\mathcal{W}} &= \{\nabla\mathcal{H}_i - \nabla\mathcal{H}_i^*\}^\top \dot{w} && \text{(30a)}\\
&= \{\nabla\mathcal{H}_i - \nabla\mathcal{H}_i^*\}^\top \{A\nabla\mathcal{H}_i && \text{(30b)}\\
&\qquad - E(B\underline{1}v_0 - d_c) - Ed_d(t)\} && \text{(30c)}\\
&= \underbrace{\{\nabla\mathcal{H}_i - \nabla\mathcal{H}_i^*\}^\top \{A\nabla\mathcal{H}_i - E(B\underline{1}v_0 - d_c)\}}_{a}\\
&\qquad - \{\nabla\mathcal{H}_i - \nabla\mathcal{H}_i^*\}^\top Ed_d(t) && \text{(30d)}\\
&\le \underbrace{-\begin{bmatrix}\nabla_p\mathcal{H}_i\\ \nabla_p\mathcal{H}_i^*\end{bmatrix}^\top \begin{bmatrix}I\\ -I\end{bmatrix}(B+D)\,[I\ -I]\begin{bmatrix}\nabla_p\mathcal{H}_i\\ \nabla_p\mathcal{H}_i^*\end{bmatrix}}_{\tilde{a}}\\
&\qquad - [\nabla_p\mathcal{H}_i - \nabla_p\mathcal{H}_i^*]^\top d_d(t) && \text{(30e)}\\
&\le -[\nabla_p\mathcal{H}_i - \nabla_p\mathcal{H}_i^*]^\top d_d(t) && \text{(30f)}
\end{aligned}$$

where a is time derivative of the lyapunov function when subjected to constant disturbances only. Thus, a is substituted with $\tilde{a}$ as per Proposition 2.

(3) Continuing from (30e) [1],

$$\begin{aligned}
\dot{\mathcal{W}} \le& -\lambda_{\min}(B+D)|\nabla_p\mathcal{H}_i - \nabla_p\mathcal{H}_i^*|^2\\
&+ \frac{1}{2c}|\nabla_p\mathcal{H}_{cl} - \nabla_p\mathcal{H}_i^*|^2 + \frac{c}{2}|d_d(t)|^2.
\end{aligned} \tag{31}$$

Taking $c = \frac{1}{2\lambda_{\min}(B+D)}$ results in

$$\dot{\mathcal{W}} \le \frac{1}{4\lambda_{\min}(B+D)}|d_d(t)|^2 \tag{32}$$

Integrating (32) with respect to time results in

$$\begin{aligned}
\mathcal{W}(t) &\le \mathcal{W}(0) + \frac{1}{4\lambda_{\min}(B+D)}||d_d(t)||^2\\
&\le \mathcal{W}(0) + \frac{1}{4\lambda_{\min}(B+D)}\bar{d}^2,
\end{aligned} \tag{33}$$

Under the continuous change of coordinates

$$\tilde{w} := (p, \Delta, z) = (p, \Delta, p - \zeta), \tag{34}$$

the Lyapunov function (25) can be expressed as

$$\begin{aligned}
\tilde{\mathcal{W}} =& \frac{1}{2}(p - M\underline{1}v_0)^\top M^{-1}(p - M\underline{1}v_0)\\
&+ \sum_{i=1}^{N}\int_{\Delta_i^*}^{\Delta_i} f_i^s(v)dv + \frac{1}{2}k\left(z + \frac{1}{k}(B\underline{1}v_0 - d_c)\right)^\top\\
&\times (B+D)^{-1}\left(z + \frac{1}{k}(B\underline{1}v_0 - d_c)\right),
\end{aligned} \tag{35}$$

with the new equilibrium

$$\tilde{w}^* := (p^*, \Delta^*, z^*) = (M\underline{1}v_0, 0, -\frac{1}{k}B\underline{1}v_0 + \frac{1}{k}d_c), \tag{36}$$

where we have used Lemma 4. We will now show that the system, expressed in the $\tilde{w}$ coordinates, is l_2 weak string stability. As the transformation from w to $\tilde{w}$ is continuous, this is sufficient to determine l_2 weak string stability of the original system.

The Lyapunov function (35) satisfies

$$\begin{aligned}
\tilde{\mathcal{W}} \ge& \frac{1}{2}\lambda_{\min}(M^{-1})|p - M\underline{1}v_0|^2 + \frac{1}{2}L_{\min}|\Delta|^2\\
&+ \frac{1}{2}k\lambda_{\min}(B+D)^{-1}\left|z + \frac{1}{k}(B\underline{1}v_0 - d_c)\right|^2\\
\ge& \underline{s}|\tilde{w} - \tilde{w}^*|^2,
\end{aligned} \tag{37}$$

where

$$\underline{s} = \min\left\{\frac{\lambda_{\min}(M^{-1})}{2}, \frac{L_{\min}}{2}, \frac{k\lambda_{\min}(B+D)^{-1}}{2}\right\}, \tag{38}$$

and L_{min} is the minimum Lipschitz constant for all f_i on the domain of the system. Likewise, (35) also satisfies

$$\tilde{\mathcal{W}} \le \bar{s}|\tilde{w} - \tilde{w}^*|^2, \tag{39}$$

where

$$\bar{s} = \max\left\{\frac{\lambda_{\max}(M^{-1})}{2}, \frac{L_{\max}}{2}, \frac{k\lambda_{\max}(B+D)^{-1}}{2}\right\}, \tag{40}$$

and L_{max} is the maximum Lipschitz constant for all f_i on the domain of the system. As $\mathcal{W}(p(t), \Delta(t), \zeta(t)) = \tilde{\mathcal{W}}(p(t), \Delta(t), z(t))$, we utilise the inequality (33), together with (37) and (39) to find

$$\begin{aligned}
\sqrt{\underline{s}}|\tilde{w}(t) - \tilde{w}^*| &\le \sqrt{\tilde{\mathcal{W}}(t)}\\
&\le \sqrt{\tilde{\mathcal{W}}(0) + \frac{1}{4\lambda_{\min}(B+D)}\bar{d}^2}\\
&\le \sqrt{\tilde{\mathcal{W}}(0)} + \frac{1}{2\sqrt{\lambda_{\min}(B+D)}}\bar{d}\\
&\le \sqrt{\bar{s}}|\tilde{w}(0) - \tilde{w}^*| + \frac{1}{2\sqrt{\lambda_{\min}(B+D)}}\bar{d},
\end{aligned} \tag{41}$$

which can be further simplified to

$$\begin{aligned}
|\tilde{w}(t) - \tilde{w}^*| &\le \sqrt{\frac{\bar{s}}{\underline{s}}}|\tilde{w}(0) - \tilde{w}^*| + \frac{1}{2\sqrt{\underline{s}\lambda_{\min}(B+D)}}\bar{d}\\
&< \sqrt{\frac{\bar{s}}{\underline{s}}}\delta_1(\epsilon) + \frac{1}{2\sqrt{\underline{s}\lambda_{\min}(B+D)}}\delta_2(\epsilon).
\end{aligned} \tag{42}$$

Taking

$$\begin{aligned}
\delta_1 &= \epsilon\frac{1}{2}\sqrt{\frac{\underline{s}}{\bar{s}}}\\
\delta_2 &= \epsilon\sqrt{\underline{s}\lambda_{\min}(B+D)}
\end{aligned} \tag{43}$$

implies that

$$|\tilde{w}(t) - \tilde{w}^*| < \epsilon \tag{44}$$

as desired. As $B > 0$, $\lambda_{\min}(B+D)$ is bounded from below, independent of N, due to Gershgorin's Theorem. Thus, provided that $|\tilde{w}(0) - \tilde{w}^*|$ and $\bar{d}$ do not grow with N, the system is l_2 weakly string stable.

5. SIMULATION

The system was simulated for two string lengths, $N = 10$ and $N = 100$. The parameters used for simulation were

[1] Given two vectors $X, Y \in \mathbb{R}^n$, it can be verified that they satisfy $X^\top Y \le \frac{c}{2}X^\top X + \frac{1}{2c}Y^\top Y$, for any positive constant c, by expanding the inequality $(cY - X)^\top(cY - X) \ge 0$ and solving for $X^\top Y$.

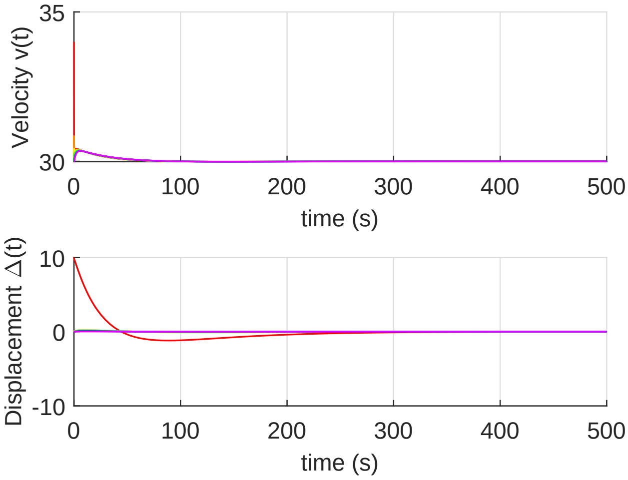

Fig. 1. Velocities and displacements of a vehicle string with $N = 10$. The string is initialised with an error in the first vehicles position and velocity. The first vehicle is denoted by the red line whereas the last is denoted by purple.

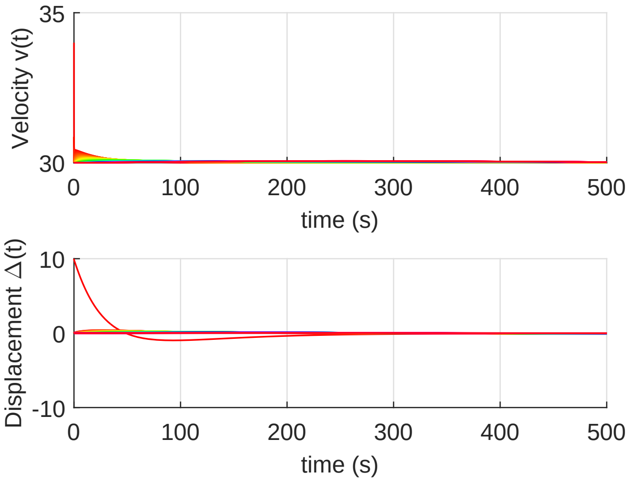

Fig. 2. Velocities and displacements of a vehicle string with $N = 100$. The string is initialised with an error in the first vehicles position and velocity. The first vehicle is denoted by the red line whereas the last is denoted by purple.

$m_i = 1$, $D_i = 20$, $b_i = 0.1$, $k = 0.01$ and $f_i^s(\Delta_i) = \Delta_i + 0.1\Delta_i^2$ for all i. The target string velocity was chosen as $v_0 = 30$. In both simulations, the initial conditions were chosen as $\Delta_1 = 10$, $p_1 = 34$, $\Delta_j = 0$, $p_j = 30$ and $\zeta_i = m_i v_0 + \frac{1}{k} b_i v_0$ for all $j \in \{2, \cdots, N\}$, $i \in \{1, \cdots, N\}$.

The simulation results of the velocities and displacements for the $N = 10$ case can be found in Figure 1. Likewise, the velocities and displacements for the $N = 100$ case can be found in Figure 2.

In both scenarios, the string tends towards the desired operating point as expected. The time scale of the transients observed are compatible to the results of (Knorn et al., 2014). However, in contrast to (Knorn et al., 2014), the transients do not show oscillatory behaviour.

6. CONCLUSION

In this paper, we presented a method to achieve l_2 weak string stability of a vehicle platoon that does not require communications within the string. The observed results perform compatibly with previous work, but relax the required information of the neighbour vehicles.

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