# Asymptotic Stability of Two-Dimensional Continuous Roesser Models with Singularities at the Stability Boundary

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***Abstract*— It is shown that the existence of a negative *semi*definite solution $Q$ of the Lyapunov equation $A^{\mathrm{T}}P + AP = Q$ with a positive definite block diagonal matrix $P = P^{\mathrm{T}}$ together with simple additional conditions is sufficient to guarantee asymptotic stability. The stability conditions presented can be used to study a wider range of dynamical systems, including systems with singularities at the stability boundary, which cannot be exponentially stable.**

## I. Introduction

In this work we will discuss asymptotic stability of a special class of two-dimensional systems. Here, two-dimensional refers to the fact that functions and variables depend not only on one independent variable such as time or space but on two completely independent continuous variables $t_1$ and $t_2$.

Due to a broad range of applications the related field of two-dimensional discrete systems has been studied in a more comprehensive form. One of the earliest discussions of stability of such systems was presented by Shanks *et al.* in [1]. Using the transfer function of the system in frequency domain (z-biplane), $H_z(z_1,z_2) = \mathrm{num}(z_1,z_2)/\mathrm{den}(z_1,z_2)$ they showed BIBO stability for systems devoid of poles outside the stability region. This led to different stability tests presented in the literature, such as [2], [3], and was later extended to show exponential stability in the frequency domain by [4].

Around the same time an explicit state space description was presented by Roesser in [5]. Two well known models were introduced by Fornasini and Marchesini in [6], [7].

With the appearance of state space formulations the first stability results using LMIs were published. Fornasini and Marchesini presented a sufficient stability condition for their second model in [8]. However, the first LMI based necessary and sufficient stability condition for repetitive processes with finite path length (i.e. one of the dimensions is bounded) and dynamic boundary conditions was presented by [9]. Ebihara *et al.* expanded this result for systems with unbounded dimensions in [10].

Although the second model of Fornasini–Marchesini has attracted most attention a sufficient condition for stability of the first model of Fornasini–Marchesini was developed in [11], and necessary and sufficient conditions for stability can be found in [12].

Using the state space description by Roesser, Lodge and Fahmy claimed in [13] that the characteristic polynomial $B(z_1,z_2)$ fulfills Shank's stability criterion if and only if there exists a positive definite, symmetric matrix $P = P_1 \oplus P_2$ where $P_1 \in \mathbb{R}^{n_1\times n_1}$ and $P_2 \in \mathbb{R}^{n_2\times n_2}$ and $\oplus$ denotes the direct sum, such that $A^{\mathrm{T}}PA - P = Q < 0$. However, Anderson *et al.* later showed that in general the existence of such a $P$ is sufficient but not necessary for stability [14].

For two-dimensional continuous systems some similar results have been presented in the literature. In [15] Ansell showed stability conditions using the transfer function $H_s(s_1,s_2) = A(s_1,s_2)/B(s_1,s_2)$ in the frequency (s) biplane for systems devoid of poles with non negative real parts of $s_1$ and $s_2$. This is the continuous equivalent to the condition presented by Shanks *et al.*, [2], for the discrete case. Different conditions to test for two-dimensional very strict Hurwitz polynomials have been published in [16]–[18].

A different approach to study stability of two-dimensional continuous systems based on the impulse response was taken by Jury and Bauer in [19].

As in the field of discrete systems, LMI–based stability conditions have been developed for two-dimensional continuous systems. In contrast to discrete systems, researchers have focused on the continuous version of the Roesser model. Piekarski presented in [20] a sufficient stability condition: The system is stable if there exists a positive definite, symmetric matrix $P = P_1 \oplus P_2$ such that $A^{\mathrm{T}}P + PA = Q < 0$. This was extended by Galkowski in [21].

Furthermore an important special case is often excluded in the stability discussions mentioned above. This is the case when there exists a set of $(z_1,z_2)$ (in the discrete-discrete case) or $(s_1,s_2)$ (in the continuous-continuous case) such that both the denominator and the numerator of the transfer function go to zero at the same time. In contrast to the case where the numerator is non-zero for $(z_1,z_2)$ or $(s_1,s_2)$ (nonessential singularity of the first kind) these special points are often called nonessential singularities of the second kind (NSSK). Note that the state space matrix $A$ of every system with a NSSK at the stability boundary will therefore also exhibit a singularity at the stability boundary (SSB). Although most researchers try to avoid these special cases, they sometimes cannot be avoided due to special needs in the application studied or they are even desirable to obtain a system with special properties, [22].

Goodman showed in [23] that some transfer functions with NSSK are BIBO stable, while some with NSSK at the same point in the biplane are BIBO unstable. A sufficient BIBO stability condition in the frequency domain for two-

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dimensional discrete systems with NSSK at the boundary of the bidisc (i.e. $|z_1| = |z_2| = 1$) has been presented in [24]. This was followed by a necessary condition that stability can only be achieved when the NSSK occur at the border of the bidisc in [25].

Although these results were obtained in the frequency domain, it should be noted that previous LMI–based results in time domain also exclude systems with SSB since a sign definite solution of the LMI is required. However, as we will show later, a system including a SSB cannot achieve a sign definite solution to the required LMI. The LMI–based conditions for asymptotic stability of two-dimensional continuous systems presented in this paper only require a semidefinite solution and are therefore suitable to discuss stability of systems with SSB.

Before presenting our main theorem in Section IV we will clarify our notation in Section II and give mathematical preliminaries in Section III. The paper closes with illustrative examples in Section V and conclusions in Section VI.

## II. Notation

Consider the following autonomous two-dimensional continuous Roesser model

$$\begin{pmatrix} \frac{\partial}{\partial t_1} x_1(t_1,t_2) \\ \frac{\partial}{\partial t_2} x_2(t_1,t_2) \end{pmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} x_1(t_1,t_2) \\ x_2(t_1,t_2) \end{pmatrix}}_{x(t_1,t_2)} \tag{1}$$

with the initial or boundary conditions

$$x_{1_0}(t_2) = x_1(0,t_2) \qquad \text{and} \qquad x_{2_0}(t_1) = x_2(t_1,0) \tag{2}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, and the real matrix $A$ and its submatrices are chosen with the appropriate dimensions. We will discuss the stability of such system according to the following definitions.

*Definition 1 ($L_2$ and $L_\infty$ Bounded Initial Conditions):* The initial conditions of a two-dimensional continuous Roesser Model are $L_2$ bounded, if there exist $c_i < \infty$ such that for $i \in \{1,2\}$:

$$\|x_{i_0}(\cdot)\|_2^2 = \int_0^\infty x_{i_0}^{\mathrm{T}}(t) x_{i_0}(t) \mathrm{d}t \leq c_i. \tag{3}$$

The initial conditions of a two-dimensional continuous Roesser Model are $L_\infty$ bounded, if there exist $\zeta_i < \infty$ such that for $i \in \{1,2\}$:

$$\|x_{i_0}(\cdot)\|_\infty = \sup_{t>0} |x_{i_0}(t)| \leq \zeta_i. \tag{4}$$

*Definition 2 ($L_2'$ and $L_\infty''$ Smooth Bounded Initial Cond.):* The initial conditions of a two-dimensional continuous Roesser Model are in $L_2'$ and $L_\infty''$ if there are $L_2$ and $L_\infty$ bounded according to Definition 1 and in addition there exist $c_i', \zeta_i', \zeta_i'' < \infty$ such that for $i \in \{1,2\}$:

$$\|\dot{x}_{i_0}(\cdot)\|_2^2 = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} x_{i_0}^{\mathrm{T}}(t) \frac{\mathrm{d}}{\mathrm{d}t} x_{i_0}(t) \mathrm{d}t \leq c_i', \tag{5}$$

$$\|\dot{x}_{i_0}(\cdot)\|_\infty = \sup_{t>0} \left| \frac{\mathrm{d}}{\mathrm{d}t} x_{i_0}(t) \right| \leq \zeta_i' \tag{6}$$

$$\|\ddot{x}_{i_0}(\cdot)\|_\infty = \sup_{t>0} \left| \frac{\mathrm{d}^2}{\mathrm{d}t^2} x_{i_0}(t) \right| \leq \zeta_i''. \tag{7}$$

*Definition 3 (Stability of 2D Continuous Roesser Model):* The autonomous two-dimensional continuous Roesser Model (1) is stable if for any $L_2$ and $L_\infty$ bounded initial conditions (according to Definition 1), there exists a constant $M < \infty$ such that for all $t_1,t_2$:

$$|x(t_1,t_2)|^2 = x^{\mathrm{T}}(t_1,t_2) x(t_1,t_2) \leq M. \tag{8}$$

*Definition 4 (Asymp. Stab. of 2D Sys. with S. B. I. C.):* The autonomous two-dimensional continuous Roesser Model (1) is asymptotically stable, if it is stable, and for $L_2'$ and $L_\infty''$ smooth bounded initial conditions (according to Definition 2) the following limit holds:

$$\lim_{t_1,t_2 \to \infty} x(t_1,t_2) = 0. \tag{9}$$

Note that asymptotic stability requires the states to tend to zero as $t_1$ and $t_2$ tend to $+\infty$ at the same time but in any possible form and direction.

## III. Mathematical Preliminaries

*Lemma 1:* Consider the autonomous two-dimensional continuous system (1). If the characteristic polynomial has a singularity at the stability boundary, i.e. $\mathrm{den}(s_1,s_2) = 0$ for $s_1 = j\omega_1$ and $s_2 = j\omega_2$, then for every block diagonal $P = P_1 \oplus P_2$, there exists a non-zero vector $v$ such that $v^{\mathrm{H}} Q v = 0$ where $Q = A^{\mathrm{T}} P + PA$.

*Proof:* The characteristic polynomial is equal to:

$$\mathrm{den}(s_1,s_2) = \det \begin{bmatrix} s_1 I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & s_2 I_{n_2} - A_{22} \end{bmatrix} \tag{10}$$

Since the system has a singularity at $s_1 = j\omega_1$, $s_2 = j\omega_2$, there exists a non-zero vector $v$ such that

$$\left( \begin{bmatrix} j\omega_1 I_{n_1} & 0 \\ 0 & j\omega_2 I_{n_2} \end{bmatrix} - A \right) v = 0 \tag{11}$$

Using (11) we can rewrite $v^{\mathrm{H}} Q v = v^{\mathrm{H}} (A^{\mathrm{T}} P + PA) v$ as

$$\begin{aligned} v^{\mathrm{H}} Q v = & v^{\mathrm{H}} \left( \begin{bmatrix} -j\omega_1 I_{n_1} & 0 \\ 0 & -j\omega_2 I_{n_2} \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} j\omega_1 I_{n_1} & 0 \\ 0 & j\omega_2 I_{n_2} \end{bmatrix} \right) v \end{aligned} \tag{12}$$

Thus, $v^{\mathrm{H}} Q v = 0$ independently of $P$. ■

Note therefore, that for a system including SSB it is not possible to find a positive definite matrix $P = P_1 \oplus P_2$ such that $Q$ is sign definite. However, the existence of a negative semi-definite $Q$ together with additional assumptions on $A$ might be sufficient for stability and even asymptotic stability.

*Lemma 2:* Consider the two-dimensional space of two continuous variables $t_1$ and $t_2$ and the two-dimensional non-negative vector field $V^{\mathrm{T}}(t_1,t_2) = (V_1(t_1,t_2), V_2(t_1,t_2))$. If the divergence of the vector field $V$ is non-positive for every $t_1$ and $t_2$, then the integral of $V_1(t_1,t_2)$ and $V_2(t_1,t_2)$ over $t_2 \in [0,T_2]$ and $t_1 \in [0,T_1]$, respectively, is bounded by the initial conditions $V_1(0,t_2)$ and $V_2(t_1,0)$, that is for all $T_1,T_2 > 0$:

$$\int_0^{T_2} V_1(T_1,t_2) \mathrm{d}t_2 \leq \int_0^{T_2} V_1(0,t_2) \mathrm{d}t_2 + \int_0^{T_1} V_2(t_1,0) \mathrm{d}t_1 \tag{13}$$

$$\int_0^{T_1} V_2(t_1,T_2) \mathrm{d}t_1 \leq \int_0^{T_2} V_1(0,t_2) \mathrm{d}t_2 + \int_0^{T_1} V_2(t_1,0) \mathrm{d}t_1. \tag{14}$$

*Proof:* To prove this lemma we will simply consider the surface integral of the divergence of $V$ over the rectangular region $[0,T_1]\times[0,T_2]$: $W(T_1,T_2) := \int_0^{T_2}\int_0^{T_1} \mathrm{div} V(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2$. Using the fundamental theorem of calculus or Gauss Divergence Theorem, it can be transformed into

$$W(T_1,T_2) = \int_0^{T_2} V_1(T_1,t_2)\mathrm{d}t_2 - \int_0^{T_2} V_1(0,t_2)\mathrm{d}t_2 + \int_0^{T_1} V_2(t_1,T_2)\mathrm{d}t_1 - \int_0^{T_1} V_2(t_1,0)\mathrm{d}t_1 \quad (15)$$

Since the divergence is non-positive for every $t_1$ and $t_2$, $W(T_1,T_2) \le 0$. Also, $V_2$ is a non-negative function of $t_1$ and $t_2$. Therefore (15) becomes (13). The bound on of the integral of $V_2(t_1,t_2)$ in (14) follows similarly. ■

We will now introduce the two-dimensional continuous Lyapunov function $V^\mathrm{T} = (V_1, V_2)$ with $V_1 = x_1^\mathrm{T} P_1 x_1$ and $V_2 = x_2^\mathrm{T} P_2 x_2$, where $P_1 = P_1^\mathrm{T} > 0$ and $P_2 = P_2^\mathrm{T} > 0$, to show that under some assumptions $x_1(t_1,t_2)$ and $x_2(t_1,t_2)$ in (1) are stable according to Definition 3.

*Corollary 1:* Consider the two-dimensional continuous autonomous system in (1). If the following conditions hold

(i) $A_{11}$ and $A_{22}$ are Hurwitz stable, and
(ii) there exist positive definite, symmetric matrices $P_1$ and $P_2$ such that $P = P_1 \oplus P_2$, and $Q = A^\mathrm{T}P + PA \le 0$,

then the system is stable according to Definition 3 and there exist $M_1, M_2 < \infty$ independent of $t_1$ and $t_2$, such that for all $t_1$ and $t_2$

$$|x_1(t_1,t_2)| \le M_1 \quad \text{and} \quad |x_2(t_1,t_2)| \le M_2. \quad (16)$$

*Proof:* Using the state space description given in (1), we can write $x_1(t_1,t_2)$ as

$$x_1(t_1,t_2) = \mathrm{e}^{A_{11}t_1} x_1(0,t_2) + \int_0^{t_1} \mathrm{e}^{A_{11}(t_1-\tau)} A_{12} x_2(\tau,t_2)\,\mathrm{d}\tau \quad (17)$$

Since $A_{11}$ is Hurwitz stable, there exist $\lambda_1 > 0$ and $K_1 < \infty$ such that we can transform (17) into

$$|x_1(t_1,t_2)| \le K_1 \mathrm{e}^{-\lambda_1 t_1} |x_1(0,t_2)| + \int_0^{t_1} K_1 \mathrm{e}^{-\lambda_1(t_1-\tau)} \|A_{12}\|\,|x_2(\tau,t_2)|\,\mathrm{d}\tau \quad (18)$$

We choose the Lyapunov function candidate $V_2(t_1,t_2) = x_2^\mathrm{T}(t_1,t_2) P_2 x_2(t_1,t_2)$ with $P_2 = P_2^\mathrm{T} > 0$. Using the definition of $V_2(t_1,t_2)$ and the Cauchy-Schwarz inequality (18) becomes

$$\begin{aligned}|x_1(t_1,t_2)| \le & K_1|x_1(0,t_2)| + \frac{K_1\|A_{12}\|}{\sigma_{\min}(P_2)} \int_0^{t_1} \mathrm{e}^{-\lambda_1(t_1-\tau)} \sqrt{V_2(\tau,t_2)}\mathrm{d}\tau \\ \le & K_1|x_1(0,t_2)| \\ & + \frac{K_1\|A_{12}\|}{\sigma_{\min}(P_2)} \sqrt{\int_0^{t_1} \mathrm{e}^{-2\lambda_1(t_1-\tau)}\mathrm{d}\tau \cdot \int_0^{t_1} V_2(\tau,t_2)\,\mathrm{d}\tau}\end{aligned} \quad (19)$$

With Lemma 2 and the fact that the initial conditions are in $L_2$ and $L_\infty$ (19) becomes

$$|x_1(t_1,t_2)| \le K_1\zeta_1 + \frac{K_1\|A_{12}\|}{\sigma_{\min}(P_2)\sqrt{2\lambda_1}} \sqrt{\|P_1\|c_1 + \|P_2\|c_2}$$

The boundedness of $x_2(t_1,t_2)$ can be shown similarly. ■

*Corollary 2:* Consider the autonomous two-dimensional continuous System in (1). If the following conditions hold

(i) the initial conditions are $L_2$ and $L_\infty$ bounded according to Definition 1,
(ii) $A_{11}$ and $A_{22}$ are Hurwitz stable, and
(iii) there exist positive definite, symmetric matrices $P_1$ and $P_2$ such that $P = P_1 \oplus P_2$, and $Q = A^\mathrm{T}P + AP \le 0$,

then there exist $\overline{M}_1, \overline{M}_2 < \infty$ independently of $t_2$ and $t_1$, respectively, such that

$$\int_0^\infty |x_1(t_1,t_2)|^2\,\mathrm{d}t_1 \le \overline{M}_1 \quad \text{and} \quad \int_0^\infty |x_2(t_1,t_2)|^2\,\mathrm{d}t_2 \le \overline{M}_2. \quad (20)$$

*Proof:* From (18) together with the fact that $(x+y)^2 \le 2(x^2+y^2)$, note that

$$\int_0^\infty |x_1(t_1,t_2)|^2\,\mathrm{d}t_1 \le 2K_1^2\Bigg(\int_0^\infty \mathrm{e}^{-2\lambda_1 t_1}|x_1(0,t_2)|^2\,\mathrm{d}t_1 + \|A_{12}\|^2 \int_0^\infty \left(\int_0^{t_1} \mathrm{e}^{-\lambda_1(t_1-\tau)}|x_2(\tau,t_2)|\mathrm{d}\tau\right)^2 \mathrm{d}t_1\Bigg) \quad (21)$$

The first term of the right hand side of (21) can be bounded by

$$2K_1^2 \int_0^\infty \mathrm{e}^{-2\lambda_1 t_1}|x_1(0,t_2)|^2\,\mathrm{d}t_1 \le \frac{K_1^2\zeta_1^2}{\lambda_1}. \quad (22)$$

With the Cauchy-Schwarz inequality the second term of the right hand side of (21) allows a bound to be calculated as

$$\begin{aligned}& 2K_1^2\|A_{12}\|^2 \int_0^\infty \left(\int_0^{t_1} \mathrm{e}^{-\lambda_1(t_1-\tau)}|x_2(\tau,t_2)|\mathrm{d}\tau\right)^2 \mathrm{d}t_1 \\ & \le \frac{2K_1^2\|A_{12}\|^2}{\lambda_1} \int_0^\infty \int_0^{t_1} \mathrm{e}^{-\lambda_1(t_1-\tau)}|x_2(\tau,t_2)|^2\mathrm{d}\tau\mathrm{d}t_1\end{aligned} \quad (23)$$

Interchanging the order of integration in (23) yields

$$\begin{aligned}& 2K_1^2\|A_{12}\|^2 \int_0^\infty \left(\int_0^{t_1} \mathrm{e}^{-\lambda_1(t_1-\tau)}|x_2(\tau,t_2)|\mathrm{d}\tau\right)^2 \mathrm{d}t_1 \\ & \le \frac{2K_1^2\|A_{12}\|^2}{\lambda_1} \int_0^\infty \int_\tau^\infty \mathrm{e}^{-\lambda_1(t_1-\tau)}|x_2(\tau,t_2)|^2\mathrm{d}t_1\mathrm{d}\tau \\ & \le \frac{2K_1^2\|A_{12}\|^2}{\lambda_1^2} \int_0^\infty |x_2(\tau,t_2)|^2\mathrm{d}\tau\end{aligned} \quad (24)$$

Taking the limit as $T_1 \to \infty$ of (14) in Lemma 2 we see that the integral in (24) is bounded independently of $t_2$. ■

To facilitate the proof of our main theorem in Section IV we will show that under suitable assumptions the first derivatives of $x$ are in $L_2[0,\infty)\times[0,\infty)$ and $L_\infty[0,\infty)\times[0,\infty)$, and the second derivatives are in $L_\infty[0,\infty)\times[0,\infty)$.

*Lemma 3:* Consider the autonomous two-dimensional continuous system in (1). If the following conditions hold

(i) the initial conditions are $L_2'$ and $L_\infty''$ smooth bounded according to Definition 2,
(ii) $A_{11}$ and $A_{22}$ are Hurwitz stable, and
(iii) there exist positive definite, symmetric matrices $P_1$, $P_2$ and $R$ such that $P = P_1 \oplus P_2$ and $Q = A^\mathrm{T}P + PA = -A^\mathrm{T}RA \le 0$

then

(a) the first derivatives of $x_1(t_1,t_2)$ and $x_2(t_1,t_2)$ are in $L_2[0,\infty) \times [0,\infty)$ and $L_\infty[0,\infty) \times [0,\infty)$, i.e. there exist $M_{ik},\overline{M}_{ik} < \infty$ such that for $i,k \in \{1,2\}$,

$$\sup_{(t_1,t_2)\in\mathbb{R}^+\times\mathbb{R}^+} \left|\frac{\mathrm{d}}{\mathrm{d}t_k} x_i(t_1,t_2)\right| \leq M_{ik} \tag{25}$$

$$\int_0^\infty \int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t_k} x_i(t_1,t_2)\right|^2 \mathrm{d}t_1\mathrm{d}t_2 \leq \overline{M}_{ik}, \quad \text{and} \tag{26}$$

(b) the second derivatives of $x_1(t_1,t_2)$ and $x_2(t_1,t_2)$ are in $L_\infty[0,\infty) \times [0,\infty)$, i.e. there exist $M_{ikl} < \infty$ such that for $i,k,l \in \{1,2\}$

$$\sup_{(t_1,t_2)\in\mathbb{R}^+\times\mathbb{R}^+} \left|\frac{\mathrm{d}^2}{\mathrm{d}t_k\mathrm{d}t_l} x_i(t_1,t_2)\right| \leq M_{ikl}. \tag{27}$$

*Proof:* (a): We will first show that $\frac{\mathrm{d}}{\mathrm{d}t_1}x_1(t_1,t_2)$ and $\frac{\mathrm{d}}{\mathrm{d}t_2}x_2(t_1,t_2)$ are in $L_\infty[0,\infty) \times [0,\infty)$ using the state space description for $\frac{\mathrm{d}}{\mathrm{d}t_1}x_1(t_1,t_2)$ in (1)

$$\left|\frac{\mathrm{d}}{\mathrm{d}t_1} x_1(t_1,t_2)\right| \leq \|A_{11}\| \cdot |x_1(t_1,t_2)| + \|A_{12}\| \cdot |x_2(t_1,t_2)| \tag{28}$$

Since $x_1(t_1,t_2)$ and $x_2(t_1,t_2)$ are bounded from Corollary 1 for all $t_1$ and $t_2$ by $M_1$ and $M_2$, $M_{11} = \|A_{11}\|M_1 + \|A_{12}\|M_2$. $M_{22}$ follows in the same way.

To show that $\frac{\mathrm{d}}{\mathrm{d}t_2}x_1(t_1,t_2)$ and $\frac{\mathrm{d}}{\mathrm{d}t_1}x_2(t_1,t_2)$ are in $L_\infty[0,\infty)\times[0,\infty)$ as well, we will transform the solution given in (17) into

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t_2} x_1(t_1,t_2)\right| \leq& K_1 \mathrm{e}^{-\lambda_1 t_1} \left|\frac{\mathrm{d}}{\mathrm{d}t_2} x_1(0,t_2)\right| \\ &+ \int_0^{t_1} K_1\mathrm{e}^{-\lambda_1\tau} \|A_{12}\| \left|\frac{\mathrm{d}}{\mathrm{d}t_2} x_2\left(t_1-\tau,t_2\right)\right| \mathrm{d}\tau \\ \leq& K_1\zeta_1' + \frac{K_1\|A_{12}\|M_{22}}{\lambda_1}\end{aligned} \tag{29}$$

To show that the first derivatives are also in $L_2[0,\infty)\times[0,\infty)$ we will use the Lyapunov function candidate $V(t_1,V_2)$ introduced above. Given the fact that $x^\mathrm{T}(t_1,t_2)Qx(t_1,t_2)$ is the divergence of $V(t_1,t_2)$ we can show with the fundamental theorem of calculus that

$$\begin{aligned}&\int_0^{T_2}\int_0^{T_1} \begin{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t_1}x_1^\mathrm{T}(t_1,t_2) & \frac{\mathrm{d}}{\mathrm{d}t_2}x_2^\mathrm{T}(t_1,t_2)\end{bmatrix} R \begin{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t_1}x_1(t_1,t_2)\\ \frac{\mathrm{d}}{\mathrm{d}t_2}x_2^\mathrm{T}(t_1,t_2)\end{bmatrix}\mathrm{d}t_1\mathrm{d}t_2 \\ &\leq \int_0^{T_2} V_1(0,t_2)\mathrm{d}t_2 + \int_0^{T_1} V_2(t_1,0)\mathrm{d}t_1\end{aligned} \tag{30}$$

Taking the limit on both sides as $T_1,T_2 \to \infty$ we see that

$$\overline{M}_{11} = \overline{M}_{22} = \frac{\|P_1\|c_1 + \|P_2\|c_2}{\sigma_{\min}(R)}. \tag{31}$$

To show the existence of $\overline{M}_{12}$ we will transform the solution given in (17) into

$$\begin{aligned}&\int_0^\infty\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t_2}x_1(t_1,t_2)\right|^2 \mathrm{d}t_1\mathrm{d}t_2 \\ &\leq 2K_1^2 \int_0^\infty\int_0^\infty \mathrm{e}^{-2\lambda_1 t_1}\left|\frac{\mathrm{d}}{\mathrm{d}t_2}x_1(0,t_2)\right|^2\mathrm{d}t_1\mathrm{d}t_2 \\ &\underbrace{+\,2K_1^2\|A_{12}\|^2\int_0^\infty\int_0^\infty \left|\int_0^{t_1}\mathrm{e}^{-\lambda_1(t_1-\tau)}\frac{\mathrm{d}}{\mathrm{d}t_2}x_2(\tau,t_2)\mathrm{d}\tau\right|^2\mathrm{d}t_1\mathrm{d}t_2}_{\alpha}.\end{aligned} \tag{32}$$

Since the initial conditions are $L_2'$ smooth the first term on the right side of (32) can be bounded by

$$2K_1^2\int_0^\infty\int_0^\infty \mathrm{e}^{-2\lambda_1 t_1}\left|\frac{\mathrm{d}}{\mathrm{d}t_2}x_1(0,t_2)\right|^2 \mathrm{d}t_1\mathrm{d}t_2 \leq \frac{2K_1^2 c_1'}{2\lambda_1}. \tag{33}$$

The second term can be transformed using the Cauchy Schwarz inequality so that it becomes

$$\begin{aligned}\alpha \leq& 2K_1^2\|A_{12}\|^2 \int_0^\infty\int_0^\infty \left(\int_0^{t_1}\mathrm{e}^{-\lambda_1(t_1-\tau)}\mathrm{d}\tau\right. \\ &\left.\cdot\int_0^{t_1}\mathrm{e}^{-\lambda_1(t_1-\tau_1)}\left|\frac{\mathrm{d}}{\mathrm{d}t_2}x_2(\tau,t_2)\right|^2\mathrm{d}\tau\right)\mathrm{d}t_1\mathrm{d}t_2.\end{aligned} \tag{34}$$

We will now solve the first inner integral and change the order of integration of the remaining part. Thus (34) becomes

$$\begin{aligned}\alpha \leq& \frac{2K_1^2\|A_{12}\|^2}{\lambda_1}\int_0^\infty\int_0^\infty\int_{\tau_1}^\infty \mathrm{e}^{-\lambda_1(t_1-\tau)}\left|\frac{\mathrm{d}}{\mathrm{d}t_2}x_2(\tau,t_2)\right|^2\mathrm{d}t_1\mathrm{d}\tau\mathrm{d}t_2 \\ \leq& \frac{2K_1^2\|A_{12}\|^2}{\lambda_1^2}\overline{M}_{22}.\end{aligned} \tag{35}$$

(b): To complete the proof we will show that the second derivatives are in $L_\infty$ $[0,\infty) \times [0,\infty)$. First the norm of the derivatives $\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}x_1(t_1,t_2)$ and $\frac{\mathrm{d}^2}{\mathrm{d}t_1\mathrm{d}t_2}x_1(t_1,t_2)$ will be considered. Taking the derivative of the first part of the state space description (1) with respect to $t_1$ yields

$$\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}x_1(t_1,t_2) = A_{11}\frac{\mathrm{d}}{\mathrm{d}t_1}x_1(t_1,t_2) + A_{12}\frac{\mathrm{d}}{\mathrm{d}t_1}x_2(t_1,t_2) \tag{36}$$

Thus $M_{111} = \|A_{11}\|M_{11} + \|A_{12}\|M_{12}$. To show that $\left|\frac{\mathrm{d}^2}{\mathrm{d}t_2^2}x_1(t_1,t_2)\right|$ is bounded, follow a similar argument as in (29), so that $M_{122}$ becomes

$$M_{122} = K_1\zeta_1'' + \frac{K_1\|A_{12}\|M_{222}}{\lambda_1}. \tag{37}$$

The existence of $M_{112}, M_{121}, M_{211}, M_{212}, M_{221}$ and $M_{222}$ can be proven in the same manor. ■

*Lemma 4:* Consider the two-dimensional function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. If $f(t_1,t_2)$ is both in $L_p$ $[0,\infty) \times [0,\infty)$ and $L_\infty$ $[0,\infty)\times[0,\infty)$ and both its derivatives $\frac{\mathrm{d}}{\mathrm{d}t_1}f(t_1,t_2)$ and $\frac{\mathrm{d}}{\mathrm{d}t_2}f(t_1,t_2)$ are in $L_\infty$ $[0,\infty)\times[0,\infty)$, then $\lim_{t_1,t_2\to\infty} f(t_1,t_2) = 0$ and $f(t_1,t_2)$ is uniformly convergent in both directions, i.e. for all $\epsilon > 0$ there exists a $T(\epsilon) < \infty$ such that

$$\forall (t_1,t_2) \in \{\mathbb{R}^+ \times [T(\epsilon),\infty)\} \cup \{[T(\epsilon),\infty) \times \mathbb{R}^+\} : |f(t_1,t_2)| < \epsilon. \tag{38}$$

*Proof:* Define the supremum of $f(t_1,t_2)$ and the supremum over the maximum of both derivatives in the complete quadrant as

$$\overline{f} := \sup_{t_1,t_2\in\mathbb{R}^+\times\mathbb{R}^+} |f(t_1,t_2)| \quad \text{and} \tag{39}$$

$$\overline{f'} := \sup_{t_1,t_2\in\mathbb{R}^+\times\mathbb{R}^+} \left\{\max\left\{\left|\frac{\mathrm{d}}{\mathrm{d}t_1}f(t_1,t_2)\right|; \left|\frac{\mathrm{d}}{\mathrm{d}t_2}f(t_1,t_2)\right|\right\}\right\} \tag{40}$$

and the region $R_l$ as

$$R_l := \{[0,l+1) \times [l,l+1)\} \cup \{[l,l+1) \times [0,l)\}. \tag{41}$$

Note then that

$$\|f(\cdot,\cdot)\|^p_{L_p\ [0,\infty)\times[0,\infty)} = \sum_{l=0}^{\infty} \iint_{R_l} |f(t_1,t_2)|^p \mathrm{d}t_1 \mathrm{d}t_2 < \infty \tag{42}$$

where $\iint_{R_l} \cdot\, \mathrm{d}t_1\mathrm{d}t_2$ refers to the two-dimensional integration over the region $R_l$. Therefore,

$$\lim_{l\to\infty} \iint_{R_l} |f(t_1,t_2)|^p \mathrm{d}t_1 \mathrm{d}t_2 = 0. \tag{43}$$

Let the supremum of $f$ within $R_l$ be defined as

$$\overline{f_l} := \sup_{(t_1,t_2)\in R_l} |f(t_1,t_2)|. \tag{44}$$

Then

$$\begin{aligned}\sup_{(t_1,t_2)\in R_l} \frac{\mathrm{d}}{\mathrm{d}t_1} |f(t_1,t_2)|^p \leq & \sup_{(t_1,t_2)\in R_l} \left( p|f(t_1,t_2)|^{p-1} \left| \frac{\mathrm{d}}{\mathrm{d}t_1} f(t_1,t_2) \right| \right) \\ \leq & p\overline{f_l}^{p-1}\overline{f'}. \end{aligned} \tag{45}$$

We will now bound the double integral $\iint_{R_l} |f(t_1,t_2)|^p \mathrm{d}t_1\mathrm{d}t_2$ from below by the pyramid with height $\overline{f_l}^p$, where the base is bounded by $\frac{\overline{f_l}}{p\overline{f'}}$ or the length and width of the region $R_l$.

$$\iint_{R_l} |f(t_1,t_2)|^p \mathrm{d}t_1 \mathrm{d}t_2 \geq \frac{1}{6}\overline{f_l}^p \min\left\{ \frac{\overline{f_l}}{p\overline{f'}}; l+1 \right\} \min\left\{ \frac{\overline{f_l}}{p\overline{f'}}; 1 \right\} \tag{46}$$

We can transform (46) into

$$\iint_{R_l} |f(t_1,t_2)|^p \mathrm{d}t_1 \mathrm{d}t_2 \geq \frac{1}{6}\overline{f_l}^{p+2} \min\left\{ \frac{1}{p\overline{f'}}; \frac{l+1}{\overline{f}} \right\} \min\left\{ \frac{1}{p\overline{f'}}; \frac{1}{\overline{f}} \right\}$$

Thus

$$\overline{f_l}^{p+2} \leq 6\left(\max\left\{p\overline{f'};\overline{f}\right\}\right)^2 \iint_{R_l} |f(t_1,t_2)|^p \mathrm{d}t_1 \mathrm{d}t_2 \tag{47}$$

As $\overline{f'}$ and $\overline{f}$ are bounded $\overline{f_l}$ tends to zero as $l$ grows without bound. Hence from the definition of $\overline{f_l}$ (44), $f(t_1,t_2)$ for $(t_1,t_2) \in R_l$ tends to zero as $l$ grows without bound. ■

## IV. Asymptotic Stability

In this section we will now present our main theorem for asymptotic stability of two-dimensional continuous systems described by the Roesser model.

*Theorem 1 (Asympt. Stab. of 2D Cont. Roesser Models):* The two-dimensional continuous system (1) is asymptotically stable with smooth bounded initial conditions according to Definition 4 if the following conditions hold

(i) $A_{11}$ and $A_{22}$ are Hurwitz stable, and
(ii) there exist positive definite, symmetric matrices $P_1$, $P_2$ and $R$ such that $P = P_1 \oplus P_2$ and $Q = A^\mathrm{T}P + PA = -A^\mathrm{T}RA \leq 0$.

*Proof:* Consider the two-dimensional Lyapunov function $V(t_1,t_2)$ defined earlier and the integral of $V_1(t_1,t_2) + V_2(t_1,t_2)$ along the line $\Omega(l) := (t_1,t_2) \in \{[0,l]\times\{l\}\} \cup \{\{l\}\times[0,l]\}$ for $l \in \mathbb{R}^+$, and $l > 0$ as:

$$\begin{aligned} U(l) :=& \int_{\Omega(l)} V_1(t_1,t_2) + V_2(t_1,t_2)\mathrm{d}s \\ =& \int_0^l (V_1(t_1,l) + V_2(t_1,l))\,\mathrm{d}t_1 + \int_0^l (V_1(l,t_2) + V_2(l,t_2))\,\mathrm{d}t_2 \end{aligned}$$

Using the results in Lemma 2 and Corollary 2 we see that there exists a $C$ such that for all $l$: $U(l) \leq C$. Since the first derivatives of $x(t_1,t_2)$ with respect to $t_1$ and $t_2$ are $L_2$ bounded (Lemma 3) we can find $d_{11}(l)$, $d_{12}(l)$, $d_{21}(l)$ and $d_{22}(l)$ such that for $i \in \{1,2\}$

$$d_{i1}(l) := \sup_{t_1\leq l} \left\| \frac{\mathrm{d}}{\mathrm{d}t_1} x_i(t_1,l) \right\|, \quad d_{i2}(l) := \sup_{t_2\leq l} \left\| \frac{\mathrm{d}}{\mathrm{d}t_2} x_i(l,t_2) \right\|. \tag{48}$$

Note that $d_{11}(l) \leq \sup_{t_1\geq 0} \left\| \frac{\mathrm{d}}{\mathrm{d}t_1} x_1(t_1,l) \right\|$. Making use of the version of the Barbalat's Lemma in Lemma 4, we can conclude that the first derivatives tend to zero as $t_1,t_2 \to \infty$ and are uniformly convergent in both directions. That allows us to interchange the order of supremum and limit and thus we can conclude that

$$\begin{aligned} \lim_{l\to\infty} d_{11}(l) \leq & \lim_{l\to\infty} \sup_{t_1\geq 0} \left| \frac{\mathrm{d}}{\mathrm{d}t_1} x_1(t_1,l) \right| \\ =& \sup_{t_1\geq 0} \lim_{l\to\infty} \left| \frac{\mathrm{d}}{\mathrm{d}t_1} x_1(t_1,l) \right| \\ =& 0. \end{aligned} \tag{49}$$

It can be shown in a similar way that the limits of $d_{12}(l)$, $d_{21}(l)$, and $d_{22}(l)$ for $l \to \infty$ are 0.

Thus we can bound the derivatives of $V_i(t_1,t_2)$ with respect to $t_1$ for $i \in \{1,2\}$ by

$$\forall t_1 \leq l : \frac{\mathrm{d}}{\mathrm{d}t_1} V_i(t_1,l) \leq 2n_i d_{i1}(l)\|P_i\|M_i \tag{50}$$

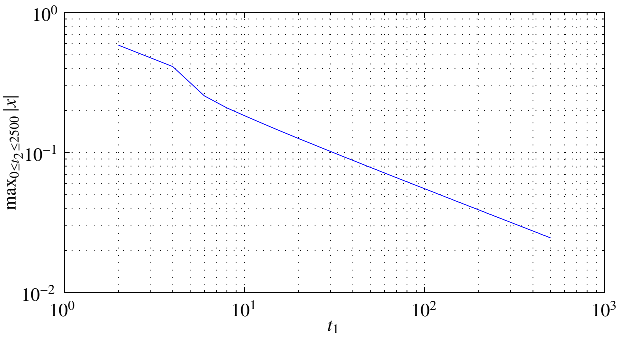
where $M_1$ and $M_2$ are bounds on $|x_1(t_1,t_2)|$ and $|x_2(t_1,t_2)|$ from Corollary 1 and $n_1$ and $n_2$ are the dimensions of $x_1(t_1,t_2)$ and $x_2(t_1,t_2)$, respectiveley. A similar bound on the derivatives with respect to $t_2$ can be found.

To find a lower bound on $U(l)$ we will use a similar trick as in the proof of Lemma 4 above. If the maximum of $V_i(t_1,t_2)$ for $i \in \{1,2\}$) along $\Omega(l)$, $\overline{V}_i(l) := \max_{(t_1,t_2)\in\Omega(l)} V_i(t_1,t_2)$, occurs along the part of $\Omega(l)$ where $(t_1,t_2) \in [0,l]\times\{l\}$ we can bound the integral of $V_i(t_1,t_2)$ over $\Omega(l)$ from below by a triangle with the base equal to $\min\left\{\overline{V}_i(l) \big/ (2n_i d_{i1}(l)\|P_i\|M_i), l\right\}$ and $\overline{V}_i(l)$ as the height of the triangle.

$$\begin{aligned} U(l) \geq & \min\left\{ \frac{\overline{V}_1^2(l)}{4n_1 d_{11}(l)\|P_1\|M_1}; \frac{\overline{V}_1^2(l)}{4n_1 d_{12}(l)\|P_1\|M_1}; \frac{\overline{V}_1(l)l}{2} \right\} \\ & + \min\left\{ \frac{\overline{V}_2^2(l)}{4n_2 d_{21}(l)\|P_2\|M_2}; \frac{\overline{V}_2^2(l)}{4n_2 d_{22}(l)\|P_2\|M_2}; \frac{\overline{V}_2(l)l}{2} \right\}. \end{aligned} \tag{51}$$

Since $\overline{V}_i(l) \leq M_i^2\|P_i\|$ this implies

$$\overline{V}_i^2(l) \leq C \cdot \max\left\{ 4n_i d_{i1}(l)\|P_i\|M_i; 4n_i d_{i2}(l)\|P_i\|M_i; \frac{2M_i^2\|P_i\|}{l} \right\} \tag{52}$$




Note that as $l$ tends to infinity each component of the maximum in (52) goes to zero and, hence, $\lim_{t_1,t_2\to\infty}|x_i(t_1,t_2)| = 0$. Note that the limits $\lim_{t_1\to\infty}|x_i(t_1,t_2)| = 0$ and $\lim_{t_2\to\infty}|x_i(t_1,t_2)| = 0$ exist as well. ■

## V. Examples

We will present a simple example to illustrate our main theorem. The system is described by the dynamic matrix

$$A = \left[\begin{array}{cc|c} -3/2 & -1/2 & 1 \\ 1 & 0 & 0 \\ \hline 1/4 & 1/4 & -1/2 \end{array}\right]. \tag{53}$$

Since the system contains a SSB at $s_1 = s_2 = 0$ $Q$ cannot be sign definite. Both $A_{11}$ and $A_{22}$ only have eigenvalues with negative real parts. Using

$$P_1 = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad P_2 = 8 \tag{54}$$

and $R = [4,1,0;1,4,0;0,0,16]$ ($\text{rank}(R) = 3$) we see that the eigenvalues of $A^{\mathrm{T}}P + PA = -A^{\mathrm{T}}RA = Q$ are $-18.34$, $-2.66$ and 0. Hence, the system is asymptotically stable. The system has been simulated for $x_{1_0}(t_2) = (\mathrm{e}^{-t_2},0)$ and $x_{2_0}(t_1) = \mathrm{e}^{-t_1}$ for $t_1,t_2 \leq 2500$. The maximum of $|x|$ over $t_2$ can be seen in the figure above.

## VI. Conclusions

A proof of asymptotic stability of two-dimensional continuous systems using Lyapunov type arguments has been presented in this paper. It has been shown that a two-dimensional continuous system with Hurwitz stable matrices $A_{11}$ and $A_{22}$ and bounded smooth initial conditions satisfying our Lyapunov condition is asymptotically stable.

We would like to stress that our Lyapunov condition only requires two positive definite matrices $P_1$ and $P_2$, such that $Q = A^{\mathrm{T}}P + PA$ (where $P = P_1 \oplus P_2$) is negative *semi*definite. Allowing $Q$ to be singular widens the areas where such stability result can be used. For example, asymptotic stability of systems with singularities at the stability boundary (where $Q$ can never be sign definite) can be tested using the main theorem presented, based on an LMI condition.

For future research, the authors will focus to extend the results presented for two-dimensional continuous-discrete systems.

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