Two-Dimensional Analysis of String Stability of Nonlinear Vehicle Strings

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Abstract— **Stability of nonlinear 2D continuous-discrete systems is shown using Lyapunov stability theory and iISS. The proposed stability conditions are applicable with non-positive divergence of the Lyapunov function. The results are used to rigorously prove string stability of a nonlinear vehicle string with variable time headway.**

I. Introduction

In the field of coordinated systems, formation control is one of many control objectives. A group of vehicles (e.g. platoon or string) is required to follow a given reference trajectory while the vehicles keep a prescribed distance to neighbouring vehicles.

In its simplest form, platoon control requires a constant distance between the vehicles and the lead vehicle to follow a given trajectory, e.g. [1], [2]. We assume: (i) a *homogeneous* string, i.e. the dynamics of the vehicle and controller are independent of the location in the string; and, (ii) distributed control, i.e. each vehicle is equipped with a controller that aims to minimised the local position error, using only locally available data. Here we consider a unidirectional string with communication range 1, where each vehicle senses the distance towards its direct predecessor or, in case of the first vehicle, the reference.

In most cases it is easy to achieve an (asymptotically) stable string in the usual sense, i.e. small initial deviations cause small perturbations (and go to zero). However, it is well known that error signals can amplify when travelling through the string resulting in growth of the local error norm with the position in the string. This effect is referred to as string instability', e.g. in [3]–[5], or 'slinky effect', e.g. in [6]–[9]. It has been shown that it is not possible to achieve string stability in a homogeneous string of strictly proper feedback control systems with nearest neighbour communications when using only linear systems with two integrators in the open loop and constant inter-vehicle spacing, [2], [10], independent of the particular linear controller design, [4].

Different methods have been proposed to overcome this problem. In [6] a *time headway policy* was introduced where the prescribed distance between each vehicle and its predecessor grows linearly with the velocity of the vehicle. If the time headway is chosen sufficiently large string stability can be guaranteed. This approach was later extended in [11] proposing a variable time headway that can be represented as a nonlinear two-dimensional system. In [12] string stability and performance of systems without time headway, with fixed time headway and with variable time headway were compared and analysed. However, string stability with variable time headway is only investigated locally, using the linearisation of the model instead of the nonlinear string.

Other papers analysing the string stability of nonlinear vehicle strings include [8], [13], [14]. In [13] the authors prove that strings of nonlinear systems using the lead velocity, the lead acceleration and local measurements are string stable if the inputs vary sufficiently slow. In [14] a global Lipschitz condition is used to guarantee string stability of nonlinear systems with sufficiently small Lipschitz constants or "weak coupling". In [8] a string of vehicles with a detailed nonlinear model and bidirectional nonlinear control is considered.

Methods used to analyse string stability range from using the Laplace transform (see e.g. [4], [10], [12], [13]) or the Z transform with respect to the position within the string (see e.g. [1]) to applying graph theory in [15] , approximating the string dynamics as a partial differential equation in [16] or using Lyapunov stability theory in [8].

There is yet another method to analyse string stability of homogeneous, unidirectional strings that we will propose here. The system can be modelled as a two-dimensional (2D) system, treating the position within the string k as an independent discrete variable resulting in a 2D continuous-discrete system depending on continuous time t and discrete position k.

This is a suitable reformulation of the string stability problem. String stability requires that the local errors (and possibly other states of the kth vehicle or subsystem) are bounded for all k. If a 2D system describing a vehicle platoon with the second independent variable k being the position within the string is stable, its states are bounded for all t and k. Thus the vehicle platoon is string stable in the sense that stability bounds are uniform with respect to k.

There is a huge body of work on the stability of linear 2D systems available. The best known results in the frequency domain include [17] whereas suitable 2D models in the time domain where presented by Roesser in [18] and Fornasini and Marchesini in [19] and their stability has been studied for example in [20]–[23].

However, relatively few results concerning the stability theory of general nonlinear 2D systems are known. Stability of a general nonlinear discrete 2D system of the form $\begin{pmatrix} x_1(k+1,l) \\ x_2(k,l+1) \end{pmatrix} = f(x(k,l),u(k,l),k,l)$ was first analysed in [24]. The main theorem guarantees uniform local stability if a scalar, positive definite Lyapunov function $\phi(x,k,l) = \phi_1(x_1,k,l) + \phi_2(x_2,k,l)$ exists such that

$$\phi'(x,k,l) := \phi_1(x,k+1,l) + \phi_2(x,k,l+1) \leq \phi. \qquad (1)$$

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In [25] the general discrete 2D Fornasini-Marchesini second model of the form $x(k+1,l+1) = f(x(k+1,l),x(k,l+1))$ for $k,l \geq 0$ was considered. Using the scalar Lyapunov function V (global) stability is guaranteed if it satisfies

$$V(f(x_1,x_2)) - aV(x_1) - bV(x_2) \leq 0 \tag{2}$$

for any $x_1,x_2 \in D \subseteq \mathbb{R}^n$ ($x_1,x_2 \in \mathbb{R}^n$), where $a,b \geq 0$ with $a+b=1$.

These results were extended in [26] to time (or parameter) varying systems of the form $x(k+1,l+1) = f(k,l,x(k+1,l),x(k,l+1))$. If there exists a Lyapunov function V satisfying

$$V(k+1,l+1,f(k,l,x_1,x_2)) - aV(k+1,l,x_1) - bV(k,l+1,x_2) \leq 0 \tag{3}$$

the system is uniformly stable.

A similar model of the form $x(k+1,l+1) = f_{01}(x(k,l+1),k,l+1) + f_{10}(x(k+1,l),k+1,l)$ for time-varying systems was studied in [27]. If a scalar positive definite function $\phi(x,k,l)$ exists such that

$$\begin{aligned}\Delta_\phi(x,k,l;\rho) &= \phi[x(k+1,l+1),k+1,l+1] \\ &-\rho\phi[x(k,l+1),k,l+1] - (1-\rho)\phi[x(k+1,l),k+1,l] \leq 0\end{aligned} \tag{4}$$

the system is uniformly stable.

It should be noted that all results available for general nonlinear 2D systems known to the authors exclusively study discrete nonlinear 2D systems.

In this work the string stability of a vehicle platoon with variable time headway shall be studied. As the 2D description of such string leads to a 2D continuous-discrete system, the development of suitable stability conditions seems necessary. The notation will be clarified in Section II before studying the stability of general nonlinear 2D continuous-discrete in Section III. Examples to illustrate our results (including the string stability discussion of a nonlinear string with variable time headway) is given in Section IV. The paper closes with concluding remarks in Section V.

II. Notation

Consider a string of N vehicles. The dynamics of the kth vehicle within the string are given by

$$\dot{z}_k(t) = \begin{cases} g_1(z_0(t),z_1(t)) & \text{for } k=1 \\ g_k(z_{k-1}(t),z_k(t)) & \text{for } 1 < k \leq N \end{cases} \tag{5}$$

where $z_k(t)$ are the local states of the kth vehicle and $z_0(t)$ is the reference signal the first vehicle within the string is aiming to follow. For simplicity we assume that $g_k(0,0) = 0$ for all k.

When assuming a homogenous string we can set $g_k = g$ for all k. Since a unidirectional string is considered, a string of length N behaves as the truncation (considering only the first N vehicles) of a string of length $M > N$ (including $M = \infty$) vehicles. Thus, a homogenous string described by (5) can be modelled as a 2D system of the form

$$\begin{pmatrix} \dot{x}_1(t,k) \\ \Delta x_2(t,k) \end{pmatrix} = \begin{pmatrix} f_1(x_1(t,k),x_2(t,k)) \\ f_2(x_1(t,k),x_2(t,k)) \end{pmatrix} \tag{6}$$

with initial (or boundary) conditions $x_{10}(k) = x_1(0,k)$ and $x_{20}(t) = x_2(t,0)$ where $x_1(t,k)$ is the first part of the state vector $x(t,k)$ of the 2D model containing the local variables of the kth vehicle (such as its position, velocity and controller states; denoted as $z_k(t)$ before). The second part of the state vector contains the information of the preceding vehicle needed for the local controller. (When the local controller aims to follow the position of the preceding vehicle, $x_2(t,k)$ will be the position of vehicle $k-1$.) Note that states and functions of the 2D system description are denoted by $x(t,k)$ and $f(t,k)$ instead of the variables and functions used for the distributed 1D system $z_k(t)$ and $g_k(t)$.

The derivative with respect to time t of x is denoted by $\dot{x}(t,k) = \frac{d}{dt}x(t,k)$. Thus, $\dot{x}_1(t,k) = f_1(x_1(t,k),x_2(t,k)) = g_k(z_{k-1}(t),z_k(t)) = g(z_{k-1}(t),z_k(t))$. The difference with respect to the position k of x or V is denoted by $\Delta x = x(t,k+1) - x(t,k)$ or $\Delta V(x) = V(x(t,k+1)) - V(x(t,k))$, respectively.

Stability of the 2D continuous-discrete model will be studied using the following 2D Lyapunov function.

Definition 1 (Two-Dimensional Lyapunov Function): A 2D function $V^{\mathrm{T}} = \begin{pmatrix} V_1(x_1) & V_2(x_2) \end{pmatrix}$ is called a 2D Lyapunov function for system (6) if:

(i) $V_i(x_i)$ is a particular type of iISS-Lyapunov function for subsystem $x_i(t,k)$, that is, there exist functions $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$, positive definite functions α_i, and constants $0 \leq b_i < \infty$ such that for $i \in \{1,2\}$

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|), \tag{7}$$

$$\dot{V}_1(x_1) \leq -\alpha_1(V_1(x_1)) + b_1 V_2(x_2), \tag{8}$$

$$\Delta V_2(x_2) \leq -\alpha_2(V_2(x_2)) + b_2 V_1(x_1); \tag{9}$$

and,

(ii) the divergence of V is non-positive:

$$\operatorname{div} V = \dot{V}_1(x_1(t,k)) + \Delta V_2(x_2(t,k)) \leq 0 \tag{10}$$

for all $t,k > 0$. ∗

Note that the shorthand notation $V_i(x_i(t,k)) = V_i(x_i) = V_i(t,k)$ is used throughout the paper.

According to the definitions of iISS-Lyapunov functions in [28] and [29] $V_1(x_1)$ needs to be continuously differentiable since t is continuous and $V_2(x_2)$ merely needs to be continuous since k is discrete. The definitions for iISS-Lyapunov functions from [28] and [29] have been altered in the way that the last term in (8) and (9) explicitly contain $b_1 V_2(x_2)$ and $b_2 V_1(x_1)$ instead of general class $\mathcal{K}_\infty$ functions $\gamma_1(x_2)$ and $\gamma_2(x_1)$.

It should also be noted similar to the stability conditions known in the literature the divergence of V, (10), only needs to be non-positive. (Compare (1), (2), (3) and (4) with (10).)

The following class of initial (or boundary) conditions will be considered.

Definition 2 (L_V and L_∞ Bounded Initial Conditions): Given positive definite functions V_i, the initial conditions of the nonlinear two-dimensional system (6) are L_V and L_∞

bounded, if there exist $c_i,\zeta_i < \infty$ for $i \in \{1,2\}$ such that

$$\|x_{10}(\cdot)\|_V := \sum_{k=0}^{\infty} V_1\left(x_{10}(k)\right) \leq c_1, \tag{11}$$

$$\|x_{20}(\cdot)\|_V := \int_0^{\infty} V_2\left(x_{20}(t)\right) \mathrm{d}t \leq c_2, \tag{12}$$

$$\|x_{10}(\cdot)\|_\infty = \sup_{k>0} |x_{10}(k)| \leq \zeta_1 \quad \text{and} \tag{13}$$

$$\|x_{20}(\cdot)\|_\infty = \sup_{t\geq 0} |x_{20}(t)| \leq \zeta_2 \tag{14}$$

is satisfied. *

Stability of 2D nonlinear continuous-discrete systems will be studied according to the following definition.

Definition 3 (Stability of Nonlinear 2D Systems): The autonomous nonlinear 2D system (6) is stable if for each $M > 0$ there exists a set of $c_i(M),\zeta_i(M) > 0$ such that if the initial conditions are L_V and L_∞ bounded with bounds $c_i(M)$ and $\zeta_i(M)$ for $i \in \{1,2\}$, respectively, then

$$|x(t,k)| \leq M \tag{15}$$

for all $t,k > 0$. *

Note that this definition of 2D stability is used here to study "string stability" of the underlying distributed 1D system. If the states are bounded in the 2D sense for all t and k for L_V and L_∞ bounded initial conditions, the local error norm of each vehicle in the underlying 1D string is bounded at all times independently of the position within the string or the string length. Thus, we study string stability with respect to nonzero, vanishing initial conditions. Note that assuming nonzero initial conditions for the leading vehicle is equivalent to studying a constant step disturbance onto the lead vehicle.

III. Stability of Nonlinear Two-Dimensional Systems

Having clarified the notation, we now give sufficient conditions for stability of general nonlinear 2D continuous-discrete systems. Two preliminary results will be given before presenting the main theorem later in this section. The first lemma was proposed in [28, Corollary IV.3].

Lemma 1 (Corollary IV.3 in [28]): Given any continuous positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there exists a $\mathcal{KL}$-function β with the following property. For any $0 < \tilde{t} \leq \infty$, and for any (locally) absolutely continuous function $V : [0,\tilde{t}) \to \mathbb{R}_{\geq 0}$ and any measurable, locally essentially bounded function $\gamma : [0,\tilde{t}) \to \mathbb{R}_{\geq 0}$, if

$$\dot{V}(t) \leq -\alpha(V(t)) + \gamma(t) \tag{16}$$

holds for almost all $t \in [0,\tilde{t})$, then the following estimate holds

$$V(t) \leq \beta(V(0),t) + \int_0^t 2\gamma(s)\mathrm{d}s \tag{17}$$

for all $t \in [0,\tilde{t})$. •

The analogous version for discrete systems has not been published explicitly as a separate lemma but can be found in [29, Proof of Theorem 2, p. 301]. It could be stated as

Lemma 2: Given any continuous positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there exists a $\mathcal{KL}$-function β with the following property. For any $0 < \tilde{k} \leq \infty$, and for any (locally) absolutely continuous function $V : \left[0,\tilde{k}\right) \to \mathbb{R}_{\geq 0}$ and any measurable, locally essentially bounded function $\gamma : \left[0,\tilde{k}\right) \to \mathbb{R}_{\geq 0}$, if

$$\Delta V(k) \leq -\alpha(V(k)) + \gamma(k) \tag{18}$$

holds for almost all $k \in \left[0,\tilde{k}\right)$, then the following estimate holds

$$V(k) \leq \beta(V(0),k) + \sum_{s=0}^{k} 2\gamma(s) \tag{19}$$

for all $k \in \left[0,\tilde{k}\right)$. •

A further result needed for the proof of stability of general nonlinear 2D continuous-discrete systems was published in a general form in [30, Lemma 4.2].

Lemma 3: Consider the 2D space of two variables t and k and the 2D non-negative vector field $V^{\mathrm{T}}(t,k) = (V_1(t,k),V_2(t,k))$. If the divergence of the vector field $V(t,k)$ is non-positive for all t and k, then

$$\sum_{l=0}^{k} V_1(t,l) \leq \sum_{l=0}^{k} V_1(0,l) + \int_0^t V_2(\tau,0)\mathrm{d}\tau \tag{20}$$

$$\int_0^t V_2(\tau,k)\mathrm{d}\tau \leq \sum_{l=0}^{k} V_1(0,l) + \int_0^t V_2(\tau,0)\mathrm{d}\tau. \tag{21}$$

for all $t,k > 0$. •

Proof: To prove this lemma we will simply consider the sum of the integral of the divergence of $V(\tau,l)$ over $t \in [0,t]$ for $l \in [0,k]$:

$$W(t,k) := \sum_{l=0}^{k} \int_0^t \left(\dot{V}_1(\tau,l) + \Delta_2 V_2(\tau,l)\right)\mathrm{d}\tau. \tag{22}$$

Using the fundamental theorem of calculus or Gauss Divergence Theorem for the continuous variable τ and simple arithmetic for the discrete variables l, (22) can be transformed into

$$W(t,k) = \sum_{l=0}^{k} V_1(t,l) - \sum_{l=0}^{k} V_1(0,l) + \int_0^t V_2(\tau,k)\mathrm{d}\tau - \int_0^t V_2(\tau,0)\mathrm{d}\tau. \tag{23}$$

Since the divergence is nonpositive for every τ and l, from (22) we get $W(t,k) \leq 0$. Also, $V_2(\tau,l)$ is a nonnegative function of τ and l. Therefore (23) implies (20). The bound on of the integral of $V_2(t,k)$ in (21) follows equivalently. ■

This now enables us to state our main theorem.

Theorem 1 (Stability of Nonlinear 2D Systems): The nonlinear 2D system (6) is stable if there exists a two-dimensional Lyapunov function according to Definition 1. •

Proof: From (20)-(21) together with the fact that the initial conditions are L_V bounded we get

$$\sum_{l=0}^{k} V_1(t,l) \leq c_1 + c_2 \quad \text{and} \quad \int_0^t V_2(\tau,k)\mathrm{d}\tau \leq c_1 + c_2. \tag{24}$$

Applying Lemma 1 we can guarantee that there exists a function $\beta_1 \in \mathcal{KL}$ such that

$$V_1(x_1(t,k)) \le \beta_1(V_1(x_{10}(k)),t) + \int_0^t 2b_1 V_2(x_2(\tau,k))\mathrm{d}\tau. \quad (25)$$

Using the fact that the initial conditions are in L_∞ and (24), equation (25) yields

$$V_1(x_1(t,k)) \le \beta_1(\zeta_1,t) + 2b_1(c_1 + c_2). \quad (26)$$

Since there exists a class $\mathcal{K}_\infty$ function $\underline{\alpha}_1(|x_1|) \le V_1(x)$ we find that

$$|x_1(t,k)| \le M_1 := \underline{\alpha}_1^{-1}(\beta_1(\zeta_1,0) + 2b_1(c_1 + c_2)) < \infty \quad (27)$$

for all t,k. Note that the bound M_1 depends on the norm of the initial conditions, i.e. ζ_1,c_1,c_2. Thus, the maximal value of $|x_1|$ for all t,k is determined by the norm of the initial conditions. Furthermore, if ζ_1, c_1 and c_2 tend to zero, then M_1 also tends to zero. A similar bound $M_2 < \infty$ for the norm of x_2 can also be found and thus the system is stable according to Definition 3. ■

IV. Examples

To illustrate our main theorem we present two examples.

Example 1: Consider the continuous-discrete 2D system

$$\dot{x}_1(t,k) = -\phi^2(x_1)x_1(t,k) + \phi(x_1)x_2(t,k) \quad (28)$$

$$\Delta x_2(t,k) = \phi(x_1)x_1(t,k) - x_2(t,k) \quad (29)$$

with the bounded function $0 < \underline{\Phi} \le \phi(x_1) \le \overline{\Phi} < \infty$.

Consider the Lyapunov function

$$V = \begin{pmatrix} V_1(x_1) \\ V_2(x_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_1^2(t,k) \\ \frac{1}{2}x_2^2(t,k) \end{pmatrix}. \quad (30)$$

Thus

$$\dot{V}_1(x_1) = -\phi^2(x_1)x_1^2(t,k) + \phi(x_1)x_1(t,k)x_2(t,k), \quad (31)$$

$$\Delta V_2(x_2) = \frac{1}{2}\phi^2(x_1)x_1^2(t,k) - \frac{1}{2}x_2^2(t,k) \quad \text{and} \quad (32)$$

$$\operatorname{div} V = -\frac{1}{2}(\phi(x_1)x_1(t,k) - x_2(t,k))^2 \le 0. \quad (33)$$

Equation (31) yields

$$\begin{aligned}\dot{V}_1(x_1) = &-\frac{1}{2}\phi^2(x_1)x_1^2(t,k) - \frac{1}{2}(\phi(x_1)x_1(t,k) - x_2(t,k))^2 \\ &+ \frac{1}{2}x_2^2(t,k) \\ \le &-\frac{1}{2}\phi^2(x_1)x_1^2(t,k) + \frac{1}{2}x_2^2(t,k) \\ \le &-\underline{\Phi}^2 V_1(x_1) + V_2(x_2). \end{aligned} \quad (34)$$

Equation (32) becomes

$$\Delta V_2(x_2) \le -V_2(x_2) + \overline{\Phi}^2 V_1(x_1). \quad (35)$$

Hence, the system with L_V and L_∞ bounded initial conditions is stable and there exists an upper bound on $|x_1|$ and $|x_2|$ for all t and k. ♡

In our second nonlinear example we will study string stability of a platoon with variable time headway. A similar variable time headway with an additional upper saturation bound was proposed in [11]. Yet, string stability of the nonlinear system has not been shown analytically but its linearisation was studied and string stability was demonstrated through simulations.

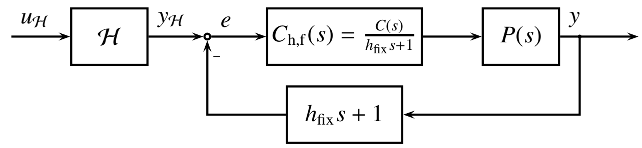

Fig. 1: Block diagram of subsystem with variable time headway

Example 2: First consider the following plant (i.e. vehicle) model

$$P(s) = \frac{1}{s^2 + 2C_\mathrm{d}v_0 s} \quad (36)$$

where the position of the kth vehicle (i.e. $y_k(t)$) is the output and the acceleration of the vehicle is used as its actuator (i.e. $u_k(t)$). The drag coefficient is given by $C_\mathrm{d} = 7 \cdot 10^{-4}$ and $v_0 = 30$ is the steady state velocity (equivalent to the reference velocity).

Each vehicle is equipped with a PID controller given by

$$C(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s} + \frac{k_\mathrm{d}s}{Ts+1} \quad (37)$$

with $k_\mathrm{p} = 1.66$, $k_\mathrm{i} = 0.17$, $k_\mathrm{d} = 4.1$ and $T = 1/30$.

The input of the local controller is the local error

$$e_k(t) = y_{k-1}(t) - y_k(t) - h_k(t)v_k(t) \quad (38)$$

where $v_k(t)$ is the velocity and $h_k(t)$ is the time headway of the kth vehicle. When a constant time headway $h_k(t) = h$ is chosen, it can be shown that the infimal time headway to guarantee string stability is $h_0 = 1.18$, [30]. Choosing a time headway $h > h_0$ guarantees that the transfer function $\Gamma(s)$ describing how the position of the kth vehicle depends on the position of its predecessor satisfies $|\Gamma(j\omega)| \le 1$ for all frequencies and $|\Gamma(j\omega)| = 1$ for $\omega = 0$ only.

Instead of a fixed time headway h here we will use a variable time headway

$$h_k(t) = h_\mathrm{var}(t,k) = h_\mathrm{fix} + h_\Delta(t,k) \quad (39)$$

where h_fix is a constant greater than the critical time headway $h_0 = 1.18$ and $h_\Delta(t,k) \ge 0$ is the variable part of the time headway. For simplicity the shorthand notation $h_\mathrm{var}(t,k) = h_\mathrm{var}$ is used below. An additional pole at $-\frac{1}{h_\mathrm{fix}}$ is added to each local controller.

In order to analyse the stability of the system we will transform the system into the scheme with the abstract block $\mathcal{H}$ in Figure 1, where the position of the kth vehicle is the input for $\mathcal{H}$ of subsystem $k+1$, i.e. $y(t,k) = u_\mathcal{H}(t,k+1)$.

We will use the following state space description for the additional state $x_{1_2}(t)$ of the system $\mathcal{H}$:

$$\dot{x}_{1_2}(t) = -\frac{1}{h_\mathrm{var}}x_{1_2}(t) + \frac{\sqrt{h_\mathrm{var} - h_\mathrm{fix}}}{h_\mathrm{var}}u_\mathrm{H}(t), \quad (40)$$

$$y_\mathrm{H}(t) = \frac{\sqrt{h_\mathrm{var} - h_\mathrm{fix}}}{h_\mathrm{var}}x_{1_2}(t) + \frac{h_\mathrm{fix}}{h_\mathrm{var}}u_\mathrm{H}(t). \quad (41)$$

Note that with h_{var} fixed, the frozen system $\mathcal{H}$ is linear, time invariant with transfer function $H(s) = \frac{h_{\text{fix}} s+1}{h_{\text{var}} s+1}$. In general, we allow h_{var} to be any time varying function that satisfies $h_{\text{var}} \geq h_{\text{fix}}$. Thus, the 2D system is described by

$$\begin{pmatrix} \dot{x}_{1_1}(t,k) \\ \dot{x}_{1_2}(t,k) \\ \Delta x_2(t,k) \end{pmatrix} = A(t,k) \begin{pmatrix} x_{1_1}(t,k) \\ x_{1_2}(t,k) \\ x_2(t,k) \end{pmatrix} \quad (42)$$

where

$$A(t,k) = \begin{bmatrix} A_0 & b_0 \sqrt{h_{\text{var}} - h_{\text{fix}}}/h_{\text{var}} & b_0 h_{\text{fix}}/h_{\text{var}} \\ 0 & -1/h_{\text{var}} & \sqrt{h_{\text{var}} - h_{\text{fix}}}/h_{\text{var}} \\ c & 0 & -1 \end{bmatrix}, \quad (43)$$

with $x_{1_1}(t,k)$ are the existing states of the controller $C(s)$ and the vehicle model $P(s)$ so that A_0 and b_0 are given by

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -2C_{\text{d}} v_0 & 1 & 0 & 0 \\ -\frac{1}{h_{\text{fix}}}\left(k_{\text{p}} + \frac{k_{\text{d}}}{T}\right) & -\left(k_{\text{p}} + \frac{k_{\text{d}}}{T}\right) & -\frac{1}{h_{\text{fix}}} & \frac{1}{h_{\text{fix}}} & -\frac{k_{\text{d}}}{h_{\text{fix}} T^2} \\ -k_{\text{i}} & -h_{\text{fix}} k_{\text{i}} & 0 & 0 & 0 \\ -1 & -h_{\text{fix}} & 0 & 0 & -\frac{1}{T} \end{bmatrix}, \quad (44)$$

$$b_0 = \begin{pmatrix} 0 & 0 & \frac{1}{h_{\text{fix}}}\left(k_{\text{p}} + \frac{k_{\text{d}}}{T}\right) & k_{\text{i}} & 1 \end{pmatrix}^{\text{T}}. \quad (45)$$

Since $x_2(t,k)$ is the position of the preceding vehicle and $x_{1_1}(t,k)$ is chosen such that its first element is the position of the kth vehicle, $c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Consider the Lyapunov function candidate V with $V_1(x_1) = x_{1_1}^{\text{T}}(t,k) P x_{1_1}(t,k) + x_{1_2}^2$ (with $P = P^{\text{T}} > 0$) and $V_2(x_2) = x_2^{\text{T}}(t,k) x_2(t,k)$. The divergence then is $x^{\text{T}} Q x$ where

$$Q = \begin{bmatrix} A_0^{\text{T}} P + P A_0 + c^{\text{T}} c & \frac{P b_0 \sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}} & \frac{P b_0 h_{\text{fix}}}{h_{\text{var}}} \\ \frac{b_0^{\text{T}} P \sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}} & -\frac{2}{h_{\text{var}}} & \frac{\sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}} \\ \frac{b_0^{\text{T}} P h_{\text{fix}}}{h_{\text{var}}} & \frac{\sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}} & -1 \end{bmatrix}. \quad (46)$$

Using the Schur complement, the requirement $Q \leq 0$ yields

$$\begin{aligned} &\begin{bmatrix} A_0^{\text{T}} P + P A_0 + c^{\text{T}} c & \frac{P b_0 \sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}} \\ \frac{b_0^{\text{T}} P \sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}} & -\frac{2}{h_{\text{var}}} \end{bmatrix} \\ &+ \begin{bmatrix} \frac{P b_0 h_{\text{fix}}}{h_{\text{var}}} \\ \frac{\sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}} \end{bmatrix} \begin{bmatrix} \frac{b_0^{\text{T}} P h_{\text{fix}}}{h_{\text{var}}} & \frac{\sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}} \end{bmatrix} \\ &= \begin{bmatrix} A_0^{\text{T}} P + P A_0 + c^{\text{T}} c + \frac{P b_0 b_0^{\text{T}} P h_{\text{fix}}^2}{h_{\text{var}}^2} & \frac{P b_0 (h_{\text{var}} + h_{\text{fix}}) \sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}^2} \\ \frac{b_0^{\text{T}} P (h_{\text{var}} + h_{\text{fix}}) \sqrt{h_{\text{var}} - h_{\text{fix}}}}{h_{\text{var}}^2} & -\frac{h_{\text{var}} + h_{\text{fix}}}{h_{\text{var}}} \end{bmatrix} \\ &\leq 0. \end{aligned} \quad (47)$$

Applying the Schur complement once again, (47) is equivalent to

$$\begin{aligned} A_0^{\text{T}} P &+ P A_0 + c^{\text{T}} c + \frac{P b_0 b_0^{\text{T}} P h_{\text{fix}}^2}{h_{\text{var}}^2} \\ &+ \frac{P b_0 b_0^{\text{T}} P (h_{\text{var}} + h_{\text{fix}})(h_{\text{var}} - h_{\text{fix}})}{h_{\text{var}}^2} \\ &= A_0^{\text{T}} P + P A_0 + c^{\text{T}} c + P b_0 b_0^{\text{T}} P \\ &\leq 0. \end{aligned} \quad (48)$$

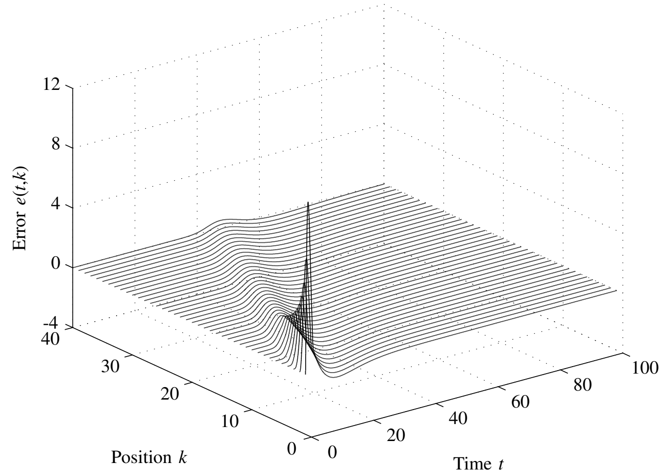

Fig. 2: String with variable time headway: error $e(t,k)$

Applying the Bounded Real Lemma we can show that existence of a $P > 0$ satisfying (48), is equivalent to the condition

$$\left\| c\,(j\omega I - A_0)^{-1}\, b_0 \right\|_\infty \leq 1. \quad (49)$$

Note that $\Gamma_0(j\omega) = c\,(j\omega I - A_0)^{-1}\, b_0$ is the transfer function from the kth to the $k+1$th vehicle for $h_{\text{var}} = h_{\text{fix}}$. Since the time headway h_{fix} is greater than the infimal time headway $h_0 = 1.18$, $|\Gamma(j\omega)| \leq 1$ for all ω and $|\Gamma(j\omega)| < 1$ for $\omega \neq 0$. Thus, a positive definite Matrix P exists such that Q is negative semi-definite independently of h_{var} (for $h_{\text{fix}} > h_0$, and $h_\Delta(x) \geq 0$) and the system is stable.

Consider the variable time headway

$$h_{\text{var}}(t,k) = \begin{cases} h_{\text{ss}} + k_{\text{h}}\,(v(t,k) - v(t,k-1)) & \text{for } h_{\text{min}} \leq h_{\text{var}}(t,k), \\ h_{\text{min}} & \text{else,} \end{cases} \quad (50)$$

where the time headway in steady state is $h_{\text{ss}} = 1.4$, $k_{\text{h}} = 0.05$ and the variable time headway is saturated at $h_{\text{fix}} = h_{\text{min}} = 1.2$. The motivation for the choice (50) is that in case the vehicle is driving slower than its predecessor, the variable time headway decreases and the vehicle thus accelerates faster and therefore can reach its desired position faster. A string of forty vehicles has been simulated. The local error is shown in Figure 2 and the variable time headway $h_{\text{var}}(t,k)$ in Figure 3.

When comparing these results to the simulation with a constant time headway of $h = 1.4$ (displayed in Figure 4) one observes that with a variable time headway with $h_{\text{ss}} = 1.4$ the error for the first vehicle increases to a maximal value that is twice as high as the maximal value of the local error of the first vehicle in a string with a constant time headway of $h = 1.4$. This is because of the decreased time headway, and consequently the desired distance between the first vehicle and reference position decreases temporarily and thus the error increases. However, with the variable time headway the local errors tend to zero quicker than choosing a constant

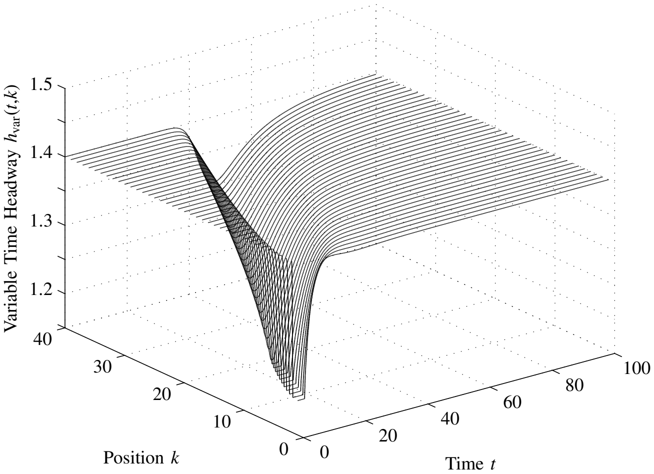

Fig. 3: String with variable time headway: $h_{\text{var}}(t,k)$

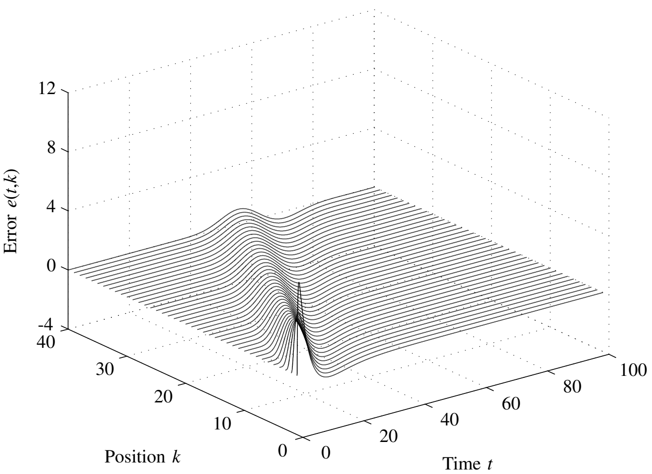

Fig. 4: String with constant time headway $h = 1.4$: $e(t,k)$

time headway. ♡

V. Conclusions

It is shown how homogeneous, unidirectional nonlinear vehicle strings can be modelled as general nonlinear 2D continuous-discrete systems. A sufficient condition for stability of general nonlinear 2D continuous-discrete systems given in a form similar to the Roesser model, [18], is presented. It is used to analytically prove string stability of a string with variable time headway.

It should be noted, however, that the methods and conditions for stability are only suitable to discuss unidirectional, homogenous strings. Also only Lyapunov type stability can be guaranteed. Additional assumptions might have to be made to ensure asymptotic stability of general nonlinear 2D continuous-discrete systems.

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