Deviation Bounds in Multi Agent Systems Described by Undirected Graphs

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Abstract

The theory of port-Hamiltonian systems is used to derive upper bounds for the state deviations in multi-agent systems described by undirected graphs pinned to a reference signal. The upper bounds for the deviations in networks of first or second order agents, respectively, depend on the minimal eigenvalue of the extended Laplacian of the system. In networks of first order agents, the deviations decay exponentially with a rate depending on the same minimal eigenvalue. In case networks of second order systems meet specific design properties, it can be shown that the deviations also decay exponentially with half the rate compared to first order systems.

Key words: Multi-agent systems, pinning control, Hamiltonian systems

1 Introduction

A Multi-Agent System (MAS) describes a group of autonomous agents operating in a networked environment. Control engineers are interested in designing strategies for a MAS to achieve global control objectives through distributed sensing, communication, computing, and control. A common control objective is "consensus" where local algorithms ensure that all agents in the system converge to the same output or state value. A simple yet robust output-feedback controller to achieve consensus is designed in Münz et al. (2011). Consensus algorithms, which are robust to time delays, network size, and modelling errors, can be found in Moreau (2004), Tian & c. L. Liu (2009), Münz et al. (2010), Das & Lewis (2010), Yang et al. (2011), Liu et al. (2010).

In the area of "pinning control", a fraction of the nodes is connected (i.e., "pinned") to a reference signal. For pinning control of networks of first-order agents see Chen et al. (2007), Ren (2007), Chen et al. (2009), Liu et al. (2009), Wang & Chen (2002). The results show that if the directed graph has a spanning tree, all agents approach a prescribed value if some are pinned. Consensus of double integrators was studied in Ren (2008). If a group reference velocity is available to each agent, then consensus is reached asymptotically if the directed interaction graph has a directed spanning tree and the gain for the velocity matching with the group reference velocity is above a certain bound. If the reference state is only available to a subset of the agents, then consensus is reached asymptotically if and only if the network is strongly connected. Lu et al. (2009) shows that linearly coupled stochastic neural networks can be controlled by a minimal number of controllers.

One of the most difficult problems in the area of pinning control is to choose the best set of pinned nodes. For scale-free networks it is much more effective to pin some highly connected nodes compared to randomly selected nodes, Wang & Chen (2002). In random networks, there is no significant difference between pinning specific or random nodes, Li et al. (2004). Yu et al. (2009) revealed that a network can realise synchronisation under any linear feedback pinning scheme by adaptively adjusting the coupling strength. V-stability was used in Xiang & Chen (2007, 2009) to develop pinning schemes. The determinants of the principle minors are used in Xiong et al. (2010) to compute which nodes should be pinned. An approach to select strongly connected components was developed in Lu et al. (2010). It was further shown in Song & Cao (2010) that nodes whose out-degrees are bigger than their in-degrees should be pinned. Further, the randomly pinning scheme may not guarantee the synchronisation of directed complex networks. Second-order nonlinear MASs were studied in Song et al. (2010).

In the area of "string stability", a group of vehicles drives in a platoon or string. In a unidirectional string, each vehicle follows its direct predecessor whereas in a bidirectional string, the distance towards the following vehicle is also used. The first vehicle follows a reference signal. This can be seen as a special case of a pinned network. Due to the simple network structure, it is trivial to ensure that all vehicles follow the trajectory. The main control objective is to design

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local controllers such that the distances between the vehicles remain bounded, independently of the string size, i.e. "string stability". It was shown in Seiler et al. (2004), Barooah & Hespanha (2005) that both unidirectional and symmetric bidirectional linear strings with two integrators in the open loop and constant spacing are always string unstable. Approaches to guarantee string stability include using: (i) a time headway, Chien & Ioannou (1992); (ii) heterogeneous controllers, Khatir & Davison (2004); (iii) information of the lead vehicle, Darbha et al. (1994); or, (iv) the reference velocity, Barooah et al. (2009). In Barooah et al. (2009), a linear bidirectional string is approximated as a PDE to derive stability bounds. This work was later extended in Hao & Barooah (2012), Hao et al. (2012). Lately, it was shown in Knorn et al. (2014) that symmetric bidirectional strings can be modelled as port-Hamiltonian systems, see van der Schaft & Jeltsema (2014), van der Schaft & Maschke (2013).

This paper extends Knorn et al. (2014) to undirected networks of single- or double-integrators, showing that:

(i) The deviations between the states and the reference signal are bounded and the upper bound depends on the smallest eigenvalues of the extended Laplacian matrix describing the pinned network, i. e. $\lambda_{\min}(\bar{\mathcal{L}})$.

(ii) In some classes of systems the deviations can be guaranteed to decay exponentially with a rate that also depends on $\lambda_{\min}(\bar{\mathcal{L}})$.

(iii) Examples are presented to illustrate the results.

Work on (i) was inspired by the problem of string stability, which aims to design local controllers ensuring the existence of a uniform bound of the inter vehicle distances. In contrast, this paper derives bounds on the deviations in general undirected graphs. Note further that our results are an extension of the well-known problem of (leader-following) consensus and pinning control. But instead of investigating under which conditions consensus can be achieved or which nodes should be pinned, it is assumed that the pinned network will converge, and the behaviour of the *deviations* towards the desired equilibrium is studied. Note that some similar results studying homogeneous systems (i.e., MAS with identical agents) have been presented in the preliminary work Hao & Barooah (2011).

Sec. 2 clarifies mathematical preliminaries. Upper bounds and decay rates for the deviations are derived in Sec. 3 and 4, respectively. Before concluding in Sec. 6, illustrative examples are presented in Sec. 5.

2 Notation and Mathematical Preliminaries

2.1 Notation

Consider the static vector $x \in \mathbb{R}^n$ and the time-varying vector $x(t) \in \mathbb{R}^n$. The L_2 vector norm is given by $|x|_2 = |x| = \sqrt{x^{\mathrm{T}}x}$ and the L_2 and L_∞ vector function norms by $\|x(\cdot)\|_2 = \sqrt{\int_0^\infty |x(t)|^2 \mathrm{d}t}$ and $\|x(\cdot)\|_\infty = \sup_{t\geq 0} |x(t)|$, respectively. For a scalar function $H(x)$ of a vector $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ its gradient is $\nabla H(x) = [\frac{\partial H(x)}{\partial x_1}, \frac{\partial H(x)}{\partial x_2}, \ldots, \frac{\partial H(x)}{\partial x_n}]^{\mathrm{T}}$. The column vector of ones is $\underline{1}$ and $\vec{e}_i \in \mathbb{R}^n$ is the ith canonical vector of length n. We denote the diagonal matrix $A \in \mathbb{R}^{n\times n}$ with diagonal entries $a_1, \ldots a_n$ as $A = \mathrm{diag}(a_1, \ldots a_n)$. Given A is symmetric positive definite ($A > 0$), $x^{\mathrm{T}}Ax \leq \lambda_{\max}(A)|x|^2$ where $\lambda_{\max}(A)$ is the maximal eigenvalue of A, Bernstein (2009). $\lambda_{\min}(A)$ denotes the minimal eigenvalue of A. The identity matrix of dimension $n \times n$ is defined as I_n. Further, $\dot{x}(t) := \frac{dx(t)}{dt}$, $\ddot{x}(t) := \frac{d^2x(t)}{dt^2}$ and "iff" = "if and only if".

2.2 Consensus Networks

In its simplest case, a consensus network consists of a group of n_{a} agents, that are simple integrators

$$\dot{x}_i(t) = u_i(t) \quad \text{for } i \in \{1,2,\ldots,n_{\mathrm{a}}\} \tag{1}$$

where $u_i(t)$ is the control input. It is the aim to reach consensus in the network, i.e., the states of each agent to converge to the weighted average of the states of its neighbours:

$$u_i(t) = \sum_{j=1, j\neq i}^{n_{\mathrm{a}}} a_{ij}(x_j(t) - x_i(t)). \tag{2}$$

where a_{ij} is the weight of the connection between agents i and j. There is no connection between i and j iff $a_{ij} = 0$.

Considering double integrator agents leads to

$$\ddot{x}_i(t) = u_i(t) \quad \text{with} \tag{3}$$

$$u_i(t) = \sum_{j=1, j\neq i}^{n_{\mathrm{a}}} a_{ij}(x_j(t) - x_i(t)) + \sum_{j=1, j\neq i}^{n_{\mathrm{a}}} r_{ij}(\dot{x}_j(t) - \dot{x}_i(t)) \tag{4}$$

where r_{ij} is the weight of the connection between the first derivatives of agents i and j. We assume $a_{ij} \neq 0$ iff $r_{ij} \neq 0$.

2.3 Graph Theory

Consider the network (3)-(4). The agents can be regarded as "nodes" or "vertices" v of a graph. An "edge" e starts at node i and ends at node j iff $a_{ij} \neq 0$. In case the input equations are symmetric, i. e. $a_{ij} = a_{ji}$ and $r_{ij} = r_{ji}$, the graph is undirected. Then, it can be described by the Laplacian matrix, which is the product of the oriented incidence matrix $\mathcal{B} \in \mathbb{R}^{n_{\mathrm{a}}\times n_{\mathrm{e}}}$ with its transpose (where n_{a} or n_{e} are the number of agents or edges, respectively), such that $\mathcal{L} = \mathcal{B}\mathcal{B}^{\mathrm{T}}$. $\mathcal{B}$ is obtained by arbitrarily choosing a direction for all e and setting $(\mathcal{B})_{ve} = 1$ if e enters v, $(\mathcal{B})_{ve} = -1$ if e leaves v and $(\mathcal{B})_{ve} = 0$ otherwise. (For an example see Sec. 5.)

2.4 Pinning Control and Reference Following

Some applications require the network to converge to a given reference. Since it is often impossible, undesirable, or unnecessary to connect all agents to the reference, only some nodes are pinned. We assume that the network is connected. Hence, pinning a single node is sufficient, Lu et al. (2009), but pinning more nodes will lead to a better performance, Patterson & Bamieh (2010). Consider (1)-(2). Pinning the first $n_{\mathrm{p}} < n_{\mathrm{a}}$ nodes to the scalar reference signal $x^*(t)$ yields

$$u_i(t) = \sum_{j=1, j\neq i}^{n_{\mathrm{a}}} a_{ij}(x_j(t) - x_i(t)) + \alpha_i(x^*(t) - x_i(t)), \tag{5}$$

for $i \leq n_\mathrm{p}$, where α_i is the weight of the connection between i and the reference. For (3)-(4), adding pinning control yields

$$u_i(t) = \sum_{j=1, j\neq i}^{n_\mathrm{a}} a_{ij}(x_j(t) - x_i(t)) + \sum_{j=1, j\neq i}^{n_\mathrm{a}} r_{ij}(\dot{x}_j(t) - \dot{x}_i(t)) + \alpha_i(x^*(t) - x_i(t)) + \rho_i(\dot{x}^*(t) - \dot{x}_i(t)), \quad (6)$$

for $i \leq n_\mathrm{p}$, where ρ_i is the weight of the connection between the first derivatives of i and the reference. Extending the graph theory above, define the extended oriented incidence matrix $\bar{\mathcal{B}} := (\vec{e}_1, \ldots, \vec{e}_{n_\mathrm{p}}, \mathcal{B})$. Following the relationship $\mathcal{L} = \mathcal{B}\mathcal{B}^\mathrm{T}$, we define the extended Laplacian as $\bar{\mathcal{L}} := \bar{\mathcal{B}}\bar{\mathcal{B}}^\mathrm{T}$.

Lemma 1 (Hong et al. (2006)) *If one or more nodes of a connected network are pinned to a reference, then* $\bar{\mathcal{L}} > 0$.

Hence, it suffices to pin a single node to ensure $\lambda_{\min}(\bar{\mathcal{L}}) > 0$. However, $\lambda_{\min}(\bar{\mathcal{L}})$ will increase when pinning more nodes, Patterson & Bamieh (2010).

2.5 Port-Hamiltonian Form

Describing (1), (2), (5) as a Hamiltonian system as in van der Schaft & Maschke (2013), van der Schaft & Jeltsema (2014) leads to

$$\dot{x}(t) = -\bar{\mathcal{B}}A\bar{\mathcal{B}}^\mathrm{T}(x(t) - \underline{1}x^*(t)) = -\bar{\mathcal{B}}A\bar{\mathcal{B}}^\mathrm{T}\frac{\partial H(x)}{\partial x} \quad (7)$$

where $A = \mathrm{diag}\{\alpha_1, \ldots \alpha_{n_\mathrm{p}}, a_1, \ldots a_{n_\mathrm{e}}\}$ with $a_e = a_{ij} = a_{ji}$ for edge e connecting i and j, $x(t) = (x_1(t), \ldots, x_{n_\mathrm{a}}(t))^\mathrm{T}$ and the Hamiltonian function [1] is given by

$$H(x(t)) = \frac{1}{2}|x(t) - \underline{1}x^*(t)|^2. \quad (8)$$

Allowing the disturbances $d(t) = (d_1(t), \ldots, d_{n_\mathrm{a}}(t))^\mathrm{T}$, leads to

$$\dot{x}(t) = -\bar{\mathcal{B}}A\bar{\mathcal{B}}^\mathrm{T}\frac{\partial H(x(t))}{\partial x} + d(t). \quad (9)$$

For second order agents (3), (4), (6), for edge e from i to j, define $\tilde{\Delta}_e(t) := x_i(t) - x_j(t)$, and for $i \leq n_\mathrm{p}$, $\hat{\Delta}_i(t) := x^*(t) - x_i(t)$. Then, for $\Delta(t) := (\hat{\Delta}_1(t), \ldots, \hat{\Delta}_{n_\mathrm{p}}(t), \tilde{\Delta}_1(t), \ldots, \tilde{\Delta}_{n_\mathrm{e}}(t))^\mathrm{T}$

$$\Delta(t) = -\bar{\mathcal{B}}^\mathrm{T}(x(t) - \underline{1}x^*(t)). \quad (10)$$

The vector of inputs $u(t) = (u_1(t), u_2(t), \ldots, u_{n_\mathrm{a}}(t))^\mathrm{T}$ yields

$$u(t) = \bar{\mathcal{B}}A\Delta(t) - \bar{\mathcal{B}}R\bar{\mathcal{B}}^\mathrm{T}(\dot{x}(t) - \dot{x}^*(t)) \quad (11)$$

with $R = \mathrm{diag}\{\rho_1, \rho_2, \ldots \rho_{n_\mathrm{p}}, r_1, r_2, \ldots r_{n_\mathrm{e}}\}$ where $r_e = r_{ij} = r_{ji}$ for edge e from i to j. Denoting $\dot{x}(t) = p(t)$, assuming a reference with $\dot{x}^*(t) = p^* = const.$ and invoking (10) and (11) yields

$$\begin{bmatrix}\dot{p}(t)\\ \dot{\Delta}(t)\end{bmatrix} = \begin{bmatrix}-\bar{\mathcal{B}}R\bar{\mathcal{B}}^\mathrm{T} & \bar{\mathcal{B}}\\ -\bar{\mathcal{B}}^\mathrm{T} & 0\end{bmatrix}\nabla H(p(t), \Delta(t)) \quad (12)$$

with the Hamiltonian function

$$H(p(t), \Delta(t)) = \frac{1}{2}(p(t) - \underline{1}p^*)^\mathrm{T}(p(t) - \underline{1}p^*) + \frac{1}{2}\Delta^\mathrm{T}(t)A\Delta(t). \quad (13)$$

Allowing the additional disturbance vector $d(t)$ leads to

$$\begin{bmatrix}\dot{p}(t)\\ \dot{\Delta}(t)\end{bmatrix} = \begin{bmatrix}-\bar{\mathcal{B}}R\bar{\mathcal{B}}^\mathrm{T} & \bar{\mathcal{B}}\\ -\bar{\mathcal{B}}^\mathrm{T} & 0\end{bmatrix}\nabla H(p(t), \Delta(t)) + \begin{pmatrix}d(t)\\ 0\end{pmatrix}. \quad (14)$$

3 Deviation Bounds in Undirected Networks

It will be shown, that the upper bound of the deviations depends on $\lambda_{\min}(\bar{\mathcal{L}})$.

Theorem 2 *Consider the system* (9) *and* (8) *with a constant reference* x^*. *Then,*

(i) the autonomous system is asymptotically stable, and

(ii) for L_2 *disturbances, the deviations are bounded by*

$$|x(t) - \underline{1}x^*|^2 \leq |x(0) - \underline{1}x^*|^2 + \frac{\|d(\cdot)\|^2}{\min_i(A_{ii})\lambda_{\min}(\bar{\mathcal{L}})}. \quad (15)$$

PROOF. $\underline{(i)}$: Using $H(x(t))$ as a Lyapunov function yields $\dot{H}(x(t)) = -(x(t) - \underline{1}x^*)^\mathrm{T}\bar{\mathcal{B}}A\bar{\mathcal{B}}^\mathrm{T}(x(t) - \underline{1}x^*) \leq 0$, which implies Lyapunov stability due to $A > 0$. Asymptotic stability follows by the invariance principle, Khalil (2001).

$\underline{(ii)}$: For (9) and setting $y(t) = x(t) - \underline{1}x^*$, similar steps as above lead to $\dot{H}(x(t)) = (x(t) - \underline{1}x^*)^\mathrm{T}(-\bar{\mathcal{B}}A\bar{\mathcal{B}}^\mathrm{T}(x(t) - \underline{1}x^*) + d(t)) \leq -\min_i(A_{ii})\lambda_{\min}(\bar{\mathcal{L}})|y(t)|^2 + y^\mathrm{T}(t)d(t)$. Then,

$$\begin{aligned}\dot{H}(x(t)) \leq & -\frac{\min_i(A_{ii})\lambda_{\min}(\bar{\mathcal{L}})}{2}|y(t)|^2 + \frac{1}{2\min_i(A_{ii})\lambda_{\min}(\bar{\mathcal{L}})}|d(t)|^2\\ & -\frac{\min_i(A_{ii})\lambda_{\min}(\bar{\mathcal{L}})}{2}\cdot\left|y(t) - \frac{1}{\min_i(A_{ii})\lambda_{\min}(\bar{\mathcal{L}})}d(t)\right|^2\\ \leq & \frac{1}{2\min_i(A_{ii})\lambda_{\min}(\bar{\mathcal{L}})}|d(t)|^2. \end{aligned} \quad (16)$$

Integrating (16) and replacing $H(x(t))$ yields (15). □

Theorem 3 *Consider the system* (14) *and* (13) *with a ramp reference such that* $\dot{x}^*(t) = p^* = const.$ *Then,*

(i) the autonomous system is asymptotically stable, and

(ii) the deviation vectors are bounded according to

$$|p(t) - \underline{1}p^*|^2 \leq |p(0) - \underline{1}p^*|^2 + \max_i(A_{ii})|\Delta(0)|^2 + \frac{\|d(\cdot)\|^2}{\min_i(R_{ii})\lambda_{\min}(\bar{\mathcal{L}})} \quad (17)$$

[1] For mechanical systems, the Hamiltonian function is usually an energy storage function describing the kinetic or potential energy of the system. Examples of Hamiltonian functions can be found in van der Schaft & Maschke (2013).

$$|\Delta(t)|^2 \leq \frac{|p(0) - \underline{1}p^*|^2 + \max_i(A_{ii})|\Delta(0)|^2 + \frac{\|d(\cdot)\|^2}{\min_i(R_{ii})\lambda_{\min}(\bar{\mathcal{L}})}}{\min_i(A_{ii})}. \tag{18}$$

PROOF. ($\underline{i}$): Using $H(p(t),\Delta(t))$ in (13) as a Lyapunov function candidate and computing its time derivative yields $\dot{H}(p(t),\Delta(t)) = -\nabla_p^{\mathrm{T}} H(p(t),\Delta(t))\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}}\nabla_p H(p(t),\Delta(t)) \leq 0$. This implies Lyapunov stability as $\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}} > 0$. Asymptotic stability follows by the invariance principle, Khalil (2001).

($\underline{ii}$): For (14), taking the derivative of $H(p(t),\Delta(t))$ and setting $y(t) = \nabla_p H(p(t),\Delta(t))$ yields $\dot{H}(p(t),\Delta(t)) \leq -\lambda_{\min}(\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}})|y(t)|^2 + y^{\mathrm{T}}d(t) \leq -\min_i(R_{ii})\lambda_{\min}(\bar{\mathcal{L}})|y(t)|^2 + y^{\mathrm{T}}d(t)$. Since $R,\bar{\mathcal{L}} > 0$, following similar steps as in (16) leads to $\dot{H}(p(t),\Delta(t)) \leq \frac{1}{2\min_i(R_{ii})\lambda_{\min}(\bar{\mathcal{L}})}|d(t)|^2$. With (13),

$$H(p(t),\Delta(t)) \leq \frac{|p(0) - \underline{1}p^*|^2}{2} + \max_i(A_{ii})\frac{|\Delta(0)|^2}{2} + \frac{\|d(\cdot)\|_2^2}{2\min_i(R_{ii})\lambda_{\min}(\bar{\mathcal{L}})} \tag{19}$$

Substituting $H(p(t),\Delta(t))$ leads to (17) and (18). □

Remark 4 (Selection of pinned nodes) *Analytical results showing the impact the pinning of specific nodes on $\lambda_{\min}(\bar{\mathcal{L}})$ are far from trivial. Hao & Barooah (2013) discussed the eigenvalues of $\bar{\mathcal{L}}$ for line graphs. Patterson & Bamieh (2010) showed that the steady state variance of the deviation is governed by the trace of $\bar{\mathcal{L}}$. Lu et al. (2010) shows that the smallest real part of eigenvalues of the Laplacian sub-matrix corresponding to the unpinned vertices can be used to measure the stabilizability of a network.*

4 Convergence Rate in Undirected Networks

The decay rate in networks of first order agents with a constant reference is bounded by $\lambda_{\min}(\bar{\mathcal{L}})$.

Theorem 5 (Wang & Chen (2002)) *Consider the system* (9) *and* (8) *with a constant reference x^*. Then, there exists a $M < \infty$ s.t. $|x_i(t) - x^*| \leq Me^{-\min_i(A_{ii})\lambda_{\min}(\bar{\mathcal{L}})t}$, $\forall i \in \{1,2,\ldots,n_a\}$.*

Similar to the first order agents case, the decay rate of some networks of second order agents is also governed by $\lambda_{\min}(\bar{\mathcal{L}})$; but the deviations decay with half the rate.

Theorem 6 *Consider the system* (12) *and* (13) *with a ramp reference such that $\dot{x}^*(t) = p^* = const.$. Assume that there exists a finite constant $\gamma > \lambda_{\min}(\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}})/4$ such that $A = \gamma R$. Then, there exists a $M < \infty$ such that $|p_i(t) - p^*| \leq Me^{-\frac{\min_i(R_{ii})\lambda_{\min}(\bar{\mathcal{L}})}{2}t}$ and $|\Delta_i(t)| \leq Me^{-\frac{\min_i(R_{ii})\lambda_{\min}(\bar{\mathcal{L}})}{2}t}$ for all $p_i(t)$ for $i \in \{1,\ldots,n_a\}$ and $\Delta_i(t)$ for $i \in \{1,\ldots,n_p + n_e\}$.*

PROOF. With $\xi(t) =: p(t) - \underline{1}p^*$, the system dynamics are

$$\begin{bmatrix}\dot{\xi}(t)\\ \dot{\Delta}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}-\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}} & \bar{\mathcal{B}}A\\ -\bar{\mathcal{B}}^{\mathrm{T}} & 0\end{bmatrix}}_{\Phi}\begin{bmatrix}\xi(t)\\ \Delta(t)\end{bmatrix}. \tag{20}$$

It is well known that the convergence rate of the system depends on the eigenvalue of Φ with the largest real part, Kailath (1980). Since $\Phi < 0$, the convergence rate is governed by the eigenvalue closest to the imaginary axis. Note that $\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}} > 0$. Assuming the algebraic multiplicity equals the geometric multiplicity for all eigenvalues of $\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}}$, denoted $\lambda_1,\ldots,\lambda_{n_a}$, there exists an invertible matrix T such that $\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}} = T\Lambda T^{-1}$ with $\Lambda = \mathrm{diag}\{\lambda_1,\ldots,\lambda_{n_a}\}$. Then,

$$\begin{bmatrix}T^{-1} & 0\\ 0 & I_{n_e+n_p}\end{bmatrix}\Phi\begin{bmatrix}T & 0\\ 0 & I_{n_e+n_p}\end{bmatrix} = \begin{bmatrix}-\Lambda & T^{-1}\bar{\mathcal{B}}A\\ -\bar{\mathcal{B}}^{\mathrm{T}}T & 0\end{bmatrix} \tag{21}$$

The roots of the polynomial $\det(sI_{n_a+n_e+n_p} - \Phi)$ are the eigenvalues of Φ, which are equivalent to the eigenvalues of the matrix in (21). Hence, using the Schur complement yields $\det(sI_{n_a+n_e+n_p} - \Phi) = \det(sI_{n_e+n_p})\det(sI_{n_a} + \Lambda + T^{-1}\bar{\mathcal{B}}A(sI_{n_e+n_p})^{-1}\bar{\mathcal{B}}^{\mathrm{T}}T) = \det(s^2 I_{n_a} + s\Lambda + T^{-1}\bar{\mathcal{B}}A\bar{\mathcal{B}}^{\mathrm{T}}T) = \det((sI_{n_a} + \frac{\Lambda}{2})^2 - \frac{\Lambda^2}{4} + T^{-1}\bar{\mathcal{B}}A\bar{\mathcal{B}}^{\mathrm{T}}T)$. It will be shown, that the eigenvalues of Φ are complex conjugate with the real part being half the eigenvalues of $-\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}}$. Setting $s = -\frac{\lambda_1}{2} + j\omega_1$ and using $A = \gamma R$ yields

$$\begin{aligned}&\det\Bigg(\left(\mathrm{diag}\left\{-\frac{\lambda_1}{2} + \frac{\lambda_1}{2},\ldots,-\frac{\lambda_1}{2} + \frac{\lambda_{n_a}}{2}\right\} + j\omega_1 I_{n_a}\right)^2\\ &\quad - \mathrm{diag}\left\{\frac{\lambda_1^2}{4},\ldots,\frac{\lambda_{n_a}^2}{4}\right\} + T^{-1}\bar{\mathcal{B}}A\bar{\mathcal{B}}^{\mathrm{T}}T\Bigg)\\ &= \det\Bigg(\mathrm{diag}\left\{0,-\frac{\lambda_1}{2} + \frac{\lambda_2}{2},\ldots,-\frac{\lambda_1}{2} + \frac{\lambda_{n_a}}{2}\right\}^2 - \omega_1^2 I_{n_a}\\ &\quad + j\omega_1\mathrm{diag}\{0,-\lambda_1 + \lambda_2,\ldots,-\lambda_1 + \lambda_{n_a}\}\\ &\quad - \mathrm{diag}\left\{\frac{\lambda_1^2}{4},\ldots,\frac{\lambda_{n_a}^2}{4}\right\} + \gamma\mathrm{diag}\{\lambda_1,\ldots,\lambda_{n_a}\}\Bigg).\end{aligned} \tag{22}$$

Since $\gamma > \frac{\lambda_{\min}(\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}})}{4}$, $\gamma\lambda_1 - \frac{\lambda_1^2}{4} > 0$. For $\omega_1 = \sqrt{\gamma\lambda_1 - \lambda_1^2/4}$ the first row in the matrix above is zero. This implies that the determinant is zero, which confirms that $-\frac{\lambda_1}{2} + j\omega_1$ is an eigenvalue of Φ. This argument can be repeated for all eigenvalues of $\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}}$. Thus, the real part of the largest eigenvalue of Φ is $-\lambda_{\min}(\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}})/2$. Using Kailath (1980) this implies that the states decay exponentially with rate $-\lambda_{\min}(\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}})/2$. Then, $\lambda_{\min}(\bar{\mathcal{B}}R\bar{\mathcal{B}}^{\mathrm{T}})) \geq \min_i(R_{ii})\lambda_{\min}(\bar{\mathcal{L}})$ yields the result. □

Remark 7 *Theorems 5 and 6 reveal that the convergence rates are governed by $\lambda_{\min}(\bar{\mathcal{L}})$. But, given the same graph structure, networks of second order agents will converge significantly slower. The minimal eigenvalue, which bounds the decay rate, is the same that also bounds the deviations as discussed in Section 3. Hence, choosing to pin a network such that $\lambda_{\min}$ is as farthest away from the imaginary axis as possible is beneficial for both important performance criteria: deviation bounds and convergence speed.*

Remark 8 *It should be noted that Theorem 6 only holds if $A = \gamma R$. In order to bound the convergence rate only*

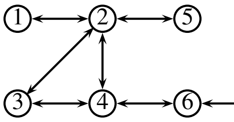

Figure 1. Example 1: Graph of Network Structure

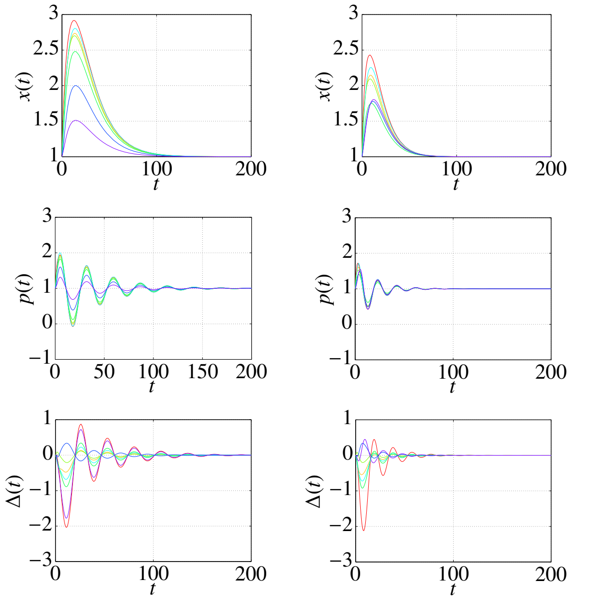

Figure 2. Example 1: $x(t)$ for first order agents; and $p(t)$ and $\Delta(t)$ for second order agents ($i = 1$ in red, $i = 2$ in orange, $\dots$, $i = n_a$ or $i = n_p + n_e$, in purple) when pinning node 7 (left) or 4 (right).

on the basis of the underlying graph, it is hence necessary to choose a fixed ratio between A and R. Further, R must be sufficiently small ensuring that the system oscillates. In order to avoid the exponentially decaying oscillations, the damping has to be chosen large enough implying that the system does not oscillate but decays with a slower rate.

5 Examples

Example 1: The first example studies the network in Fig. 1. The incidence matrix of the unpinned network is $\mathcal{B} = (1, 0, 0, 0, 0, 0, 0; -1, 1, 1, 0, 1, 0, 0; 0, -1, 0, 1, 0, 0, 0; 0, 0, -1, -1, 0, 1, 0; 0; 0, 0, 0, -1, 0, 0; 0, 0, 0, 0, 0, -1, 1; 0, 0, 0, 0, 0, 0, -1)$. The worst or best choice is to pin 7 or 4 as $\lambda_{\min}(\bar{\mathcal{L}}) = 0.0533$ or $\lambda_{\min}(\bar{\mathcal{L}}) = 0.1067$, respectively. First, consider (9), (8) with $A = I$ and $x^* = 1$. Second, consider (12) (13) with $A = R = I$ and $p^* = 1$. The effect of pinning 7 vs 4 can be observed in the simulations shown in Fig. 2: When pinning 4, the deviations are smaller and decay faster compared to pinning node 7. Further, the decay rate for first order systems is higher than in second order systems.

Example 2: The second example examines a string of 11 vehicles. In the first scenario, all vehicles are only connected to their direct neighbours with $A = R = I$ and a reference with $p^* = 1$. The first vehicle is pinned such that $\lambda_{\min}(\bar{\mathcal{L}}) = 0.018628$. $d(t)$ is a random vector times an exponentially decaying function. In the second scenario, the reference is connected to the middle vehicle. Then $\lambda_{\min}(\bar{\mathcal{L}}) =$ 0.045157. Hence, when applying the same disturbance vector, the resulting deviation are smaller compared to the first scenario and the convergence rate is improved significantly. The minimal eigenvalue of the extended Laplacian can also be increased by adding more links to the string graph: For instance, in Scenario 3, each vehicle communicates with the two preceding and the two following vehicles and first vehicle is pinned, such that $\lambda_{\min}(\bar{\mathcal{L}}) = 0.048566$. Hence, the deviation bounds and the decay rate in this scenario are similar to the second scenario. However, the third scenario requires 9 additional links. The simulation result are shown in Fig. 3.

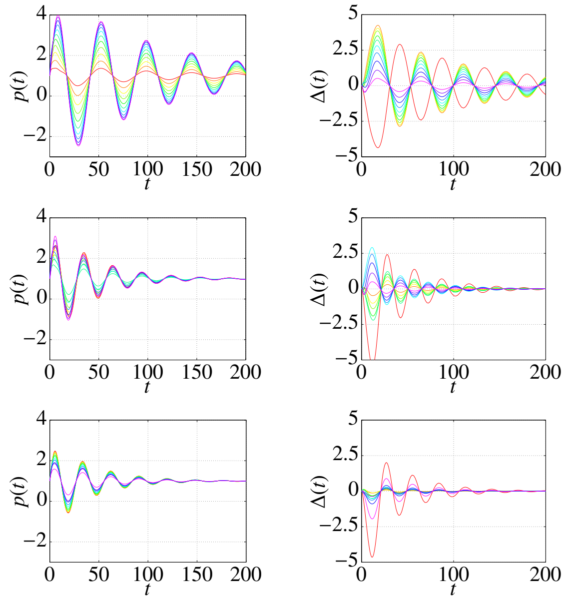

Figure 3. Example 2 : $p(t)$ and $\Delta(t)$ (as defined in (10)) ($i = 1$ in red, $i = 2$ in orange, $\dots$, $i = n_a$ or $i = n_p + n_e$, in purple) when pinning node 1 (top), node 6 (middle) or with additional connections and pinning node 1 (bottom).

6 Conclusions

We studied deviation bounds and the convergence rate of pinned undirected networks of first or second order agents. It was shown that the upper bound of the deviations from the equilibrium depends on the minimal eigenvalue of the extended Laplacian matrix of the pinned network, $\lambda_{\min}(\bar{\mathcal{L}})$. Further, it was shown that the rate of convergence in some classes of networks of second order agents also depends on $\lambda_{\min}(\bar{\mathcal{L}})$. In contrast to first order agents, the rate is halved.

Future extensions of this work should include directed networks, should aim at relaxing the assumption in Theorem 6 and investigate how to efficiently determine which nodes should be pinned to maximise $\lambda_{\min}(\bar{\mathcal{L}})$. Further, suitable local controllers to guarantee the existence of uniform deviation bounds should be designed.

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